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Publisher's version / Version de l'éditeur:
https://doi.org/10.1400/24193
MMIR MultiMedia Information Retrieval: metodologie ed esperienze internazionali di content-based retrieval per l'informazione e la documentazione, pp. 242-256, 2004

National Research Council Canada
Institute for Information Technology

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2003

* published in MMIR MultiMedia Information Retrieval: metodologie ed esperienze internazionali di content-based retrieval per l'informazione e la documentazione. ISBN 88-901144-9-5. NRC 46532.

Canada

Content-based Description of Multi-dimensional Objects using an Invariant Representation of an Associated Riemannian Space

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Abstract. This chapter presents a new theoretical approach for the description of multi-dimensional objects. These objects are characterized by various attributes such as speed, mass density and electromagnetic field distributions. The approach consists of the following steps. Firstly, a tensor is associated with the energy-momentum (mass + motion + field) content of each object. Secondly, a Riemannian space is built from this tensor. Next, a set of invariant quantities is constructed from the Riemannian curvatures associated with the Riemannian space from which a new statistical representation is built. This representation is invariant under arbitrary coordinate transformations and can describe both static and dynamic objects. The proposed approach can be generalized to a large number of different types of object by applying a variational principle.

1 Introduction

Content-based description plays a prominent role in indexation and recognition [1-3]. It is therefore important to develop compact and efficient descriptors. Up to recently, most of the efforts have been devoted to index images [1], videos [2] and three-dimensional objects [3]. Comparatively, less attention has been devoted to describe multi-dimensional objects. Each point of such an object is characterized by various physical quantities such as the mass density and the presence of a field. This chapter presents a new approach for invariant description of dynamic multi-dimensional objects under arbitrary coordinate transformations leading to a new type of histogram based on the Ricci tensor and scalar. Our study is limited to objects for which, speed, mass density and electromagnetic field distributions are known. Nevertheless, our approach is completely general and can be extended to other cases, as shown later in the conclusion. This approach is based on the transposition of certain results of general relativity [4-6] and Riemannian geometry [7] into the framework of computer vision.

The chapter is organized as follow. After some considerations on content-based description, we review the most important results of tensor analysis. Then, the fundamental covariant equations are derived from a variational principle and the energy-momentum tensor is defined. This tensor describes the mass distribution, the motion and the electromagnetic field corresponding to the object. A curved space or Riemannian space is associated with the energy-momentum tensor. Next, the geometry of the associated Riemannian space is described by the metric, the Ricci tensor and the Ricci scalar from which invariant quantities are defined. Finally, the object is described by a histogram constructed from these invariant quantities.

2 Content-based Description of Multi-dimensional Objects

An important challenge in content-based description is to find a representation that is invariant under arbitrary coordinate transformations. While popular content-based description techniques are relatively well adapted to 2D and 3D objects, their extension to multi-dimensional objects is problematic due to the high number of dimensions involved, their heterogeneity (space, time, field, speed, density, etc.) and the fact that the standard mathematical framework is not suitable to derive equations for which the form is invariant under arbitrary coordinate transformations. Form invariance is important in order to construct an object description that is invariant under arbitrary coordinate transformations. That means that no matter how an object is transformed or moved, its description is always the same. A new approach is to use tensor analysis in order to associate a Riemannian space or curved space (as opposed to flat Euclidian space) to the object. This space is described by tensorial equations invariant under arbitrary coordinate transformations from which a set of invariant quantities is extracted and new types of histograms are constructed.

3 Overview of Tensor Analysis

In this section, we present an overview of tensor algebra [7]. The use of tensorial analysis is justified by the fact that tensorial equations do not change form under arbitrary coordinate transformations. We assume that the space has four dimensions for which three are spatial and one is temporal. In other words, we treat space and time on the same level in an integrated manner. A point in 4-space is given by

(1) $$x^{\mu} \Rightarrow x = \begin{pmatrix} x^0 \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} ct \\ \mathbf{x} \end{pmatrix} \quad \text{and} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

where c is the speed of light. Unless stated otherwise, all Greek indices and all summations are to be taken from 0 to 4. Furthermore, if an index is not involved in a summation, it is immaterial and can be replaced by any other index.

We assume that it is possible to associate to the 4-space a metric $g_{\mu\nu}(x)$, which is defined by the quadratic form:

(2) $$ds^2 \equiv \sum_{\mu\nu} g_{\mu\nu}(x) dx^{\mu} dx^{\nu}$$

Indeed, because of the space curvature, it is not possible to define a global metric. It should be noticed that ds, the infinitesimal length of arc, is an invariant and has the same value irrespectively of the coordinate system. The metric defines the inner product between two tensors for the curved space. We define a covariant and a contravariant vector respectively by

(3) $$A'_{\mu}(x') = \sum_{\nu} \frac{\partial x^{\nu}(x')}{\partial x'^{\mu}} A_{\nu}(x) \quad \text{and} \quad B'^{\mu}(x') = \sum_{\nu} \frac{\partial x'^{\mu}(x)}{\partial x^{\nu}} B^{\nu}(x)$$

where

(4) $$x'^{\mu} = x'^{\mu}(x) \quad \text{and} \quad x^{\nu} = x^{\nu}(x')$$

are the arbitrary coordinate transformations. In general, a p contravariant and q covariant tensor is defined as

(5) $$T'^{\mu'_1 \ldots \mu'_p}_{\nu'_1 \ldots \nu'_q}(x') = \sum_{\mu_1 \ldots \mu_p, \nu_1 \ldots \nu_q} \frac{\partial x'^{\mu_1}(x)}{\partial x^{\mu_1}} \cdots \frac{\partial x'^{\mu_p}(x)}{\partial x^{\mu_p}} \frac{\partial x^{\nu_1}(x')}{\partial x'^{\nu'_1}} \cdots \frac{\partial x^{\nu_q}(x')}{\partial x'^{\nu'_q}} T^{\mu_1 \ldots \mu_p}_{\nu_1 \ldots \nu_q}(x)$$

We shall admit the following results without demonstration [7]. The metric is a symmetric tensor $g_{\mu\nu}(x) = g_{\nu\mu}(x)$. If a tensor is identically zero in a coordinate system, it is equal to zero in any other coordinate system. The product of a tensor by a tensor is a tensor and so is the sum. The symmetry and antisymmetry properties of a tensor are conserved under coordinate transformations. In addition, the most important property can be stated as follow: a tensorial equation does not change form under coordinate transformations. Such a feature is highly desirable if one seeks to define quantities that are coordinate transformations invariant i.e. quantities that can describe an object irrespectively of its state of motion or transformation. Furthermore, the following properties shall be of use:

(6) $$\sum_{\rho} \frac{\partial x'^{\mu}(x)}{\partial x^{\rho}} \frac{\partial x^{\rho}(x')}{\partial x'^{\nu}} = \delta_{\nu}^{\mu}$$

(7) $$g_{\mu\nu}(x) = \sum_{\rho} \frac{\partial x'^{\rho}(x)}{\partial x^{\mu}} \frac{\partial x'^{\rho}(x)}{\partial x^{\nu}}$$

(8) $$\sum_{\rho} g^{\mu\rho}(x) g_{\nu\rho}(x) = \delta_{\nu}^{\mu}$$

The metric has an additional important property; it can transform covariant indices into contravariant indices and vice versa as illustrated by the following equations:

(9) $$A_{\mu\nu} = \sum_{\alpha\beta} g_{\mu\alpha} g_{\nu\beta} A^{\alpha\beta}$$

The derivative of a tensor is not a tensor. Indeed, if one calculates the derivative of a covariant vector one obtains:

(10) $$\frac{\partial A'_{\mu}(x')}{\partial x'^{\nu}} = \sum_{\rho\sigma} \frac{\partial x^{\rho}(x')}{\partial x'^{\mu}} \frac{\partial x^{\sigma}(x')}{\partial x'^{\nu}} \frac{\partial A_{\rho}(x')}{\partial x'^{\sigma}} + \sum_{\rho} A_{\rho}(x') \frac{\partial^2 x^{\rho}(x')}{\partial x'^{\mu} \partial x'^{\nu}}$$

The first term of the right member has the correct form as defined by equations (3), but the second term is incompatible with the definition of a tensor. Nevertheless, one can define a tensorial derivative, which is covariant under coordinate transformations as follow:

(11) $$\nabla_{\nu} A_{\mu}(x) \equiv \frac{\partial A_{\mu}(x)}{\partial x^{\nu}} - \sum_{\sigma} \Gamma_{\mu\nu}^{\sigma}(x) A_{\sigma}(x)$$

One can demonstrate that the affine connection $\Gamma_{\mu\nu}^{\sigma}$ is related to the metric by

(12) $$\Gamma_{\mu\nu}^{\sigma}(x) = \sum_{\rho} \tfrac{1}{2} g^{\sigma\rho}(x) \left(\frac{\partial g_{\mu\rho}(x)}{\partial x^{\nu}} + \frac{\partial g_{\rho\nu}(x)}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}(x)}{\partial x^{\rho}} \right)$$

It should be noticed that the affine connection is not a tensor.

Riemannian spaces are not conservative: if a vector is moved along a close path, the resulting vector does not coincide in general with the original vector. That means that it is not possible to compare two tensors at two distinct positions. For an infinitesimal closed path C, one can demonstrate that the variation is proportional to the curvature tensor:

(13) $$\Delta A_\mu(x) = -\oint_C \sum_{\nu\sigma} \Gamma^\sigma_{\mu\nu}(x) A_\sigma(x) dx^\nu = \tfrac{1}{2}\sum_{\nu\sigma\tau} R^\nu_{\mu\sigma\tau}(x) A_\nu(x)\, dx^\sigma \wedge dx^\tau$$

where $\wedge$ represents the external or wedge product and where the Riemannian curvature tensor $R^\nu_{\mu\sigma\tau}(x)$ is defined by

(14) $$R^\nu_{\mu\sigma\tau}(x) \equiv \frac{\partial \Gamma^\nu_{\mu\tau}(x)}{\partial x^\sigma} - \frac{\partial \Gamma^\nu_{\mu\sigma}(x)}{\partial x^\tau} + \sum_\rho \Gamma^\nu_{\rho\sigma}(x)\Gamma^\rho_{\mu\tau}(x) - \Gamma^\nu_{\rho\tau}(x)\Gamma^\rho_{\mu\sigma}(x)$$

From the inner product between the metric and the Riemannian curvature tensor, one can define the Ricci tensor $R_{\nu\sigma}(x)$ and the Ricci scalar $R(x)$ which are respectively given by

(15) $$R_{\nu\sigma}(x) = \sum_{\mu\tau\rho} g^{\mu\rho}(x) g_{\mu\tau}(x) R^\tau_{\nu\rho\sigma}(x)$$

(16) $$R(x) = \sum_{\nu\sigma} g^{\nu\sigma}(x) R_{\nu\sigma}(x)$$

The Ricci tensor is symmetric. The Ricci tensor and scalar satisfy many identities among which are the Bianchi identities:

(17) $$\sum_\alpha \nabla_\alpha \left(R^{\sigma\alpha}(x) - \tfrac{1}{2} g^{\sigma\alpha} R(x)\right) = 0$$

4 Derivation of the Covariant Equations for the Associated Riemannian Space

In this section, we associate a Riemannian space with the object and we derive a set of tensorial equations for which the form is invariant under arbitrary coordinate transformations and from which various quantities describing the geometry of the associated space can be calculated. These quantities will be used in the next section in order to define invariant quantities.

A variational principle [5-7] allows us to formally derive our equations from a small number of hypotheses and to generalize our results as required. In order to lighten the notation, we shall not express explicitly the dependency over x unless it is suitable for intelligibility. We start from the Lagrangian, which is defined as the difference between the kinetic and the potential energy. From the Lagrangian, the action S in Riemannian space [5-7] can be defined as

(18) $$S \equiv \int\iiint d^4x \ \gamma \ L$$

where $\gamma \equiv \sqrt{-\mathbf{det}(g_{\mu\nu})}$ and $d^4x = \frac{1}{16}\sum_{\mu\nu\rho\sigma} \alpha^{\mu\nu\rho\sigma} dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma$

where $\alpha^{\mu\nu\rho\sigma}$ is the completely antisymmetric tensor [7] and L is the Lagrangian density. Consequently, the action is the integral of the Lagrangian over time. One should notice that both the Lagrangian and the action are scalar. The extra factor γ is related to the Jacobian of the transformation and ensures that the result of the integration does not depend on a particular choice of coordinate system. The principle of least action [5-7] states that if the action is extremal, the Lagrangian necessarily satisfies the Euler-Lagrange equations, which can be written in our specific case as

(19) $$\delta S = 0 \quad \Rightarrow \quad \sum_\rho \frac{\partial}{\partial x_\rho} \frac{\partial(\gamma \ L)}{\partial\left(\dfrac{\partial g_{\mu\nu}}{\partial x_\rho}\right)} - \frac{\partial(\gamma \ L)}{\partial g_{\mu\nu}} = 0$$

We are now in position to set our hypothesis and derive the corresponding equations. Let us assume that our Lagrangian can be split into two Lagrangians. The first Lagrangian $\widehat{L}$ depends solely on the metric and characterizes the Riemannian space while the second Lagrangian $\breve{L}$ depends on the metric and some other tensor Φ, which is assumed to be a function of the energy-momentum content of the object under consideration:

(20) $$S = \int\iiint d^4x \, \gamma \left[\widehat{L}(g_{\mu\nu}) + \breve{L}(g_{\mu\nu}, \Phi)\right]$$

We have indicated earlier that a Riemannian space can be characterized by a set of curvatures. One of the simplest Lagrangian that can be constructed from the Riemannian curvatures is the one constructed from the Ricci scalar:

(21) $$\widehat{L}(x) = \kappa^{-1} R = \kappa^{-1} \sum_{\mu\nu} g_{\mu\nu} R^{\mu\nu}$$

where κ is a constant. If one substitutes equation (21) into equation (20) one obtains:

(22) $$\delta S = 0 \quad \Rightarrow \quad R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R - \kappa\, \breve{T}_{\mu\nu} = 0$$

where $\breve{T}_{\mu\nu}$ is defined as the energy-momentum tensor density of the object:

(23) $$\breve{T}_{\mu\nu} \equiv \frac{\partial(\gamma\, \breve{L})}{\partial g_{\mu\nu}}$$

Because of equations (17) and (22), the energy-momentum tensor satisfies the Bianchi identities and is symmetric. Let us evaluate the energy-momentum tensor. It is well know from special relativity [6] that a Lagrangian density ${}_{>}\breve{L}$ can be associated with the motion of an object:

(24) $${}_{>}\breve{L} = \tfrac{1}{2} \rho(x) \sum_{\mu} \frac{dx^{\mu}}{d\tau} \frac{dx_{\mu}}{d\tau} = \tfrac{1}{2} \rho(x) \sum_{\alpha\mu} g_{\mu\alpha} \frac{dx^{\alpha}}{d\tau} \frac{dx^{\mu}}{d\tau} \quad \text{where} \quad d\tau \equiv ds$$

This Lagrangian density is essentially the relativistic kinetic energy density: $\rho(x)$ being the mass density and τ the proper time. If the speed of the object is much smaller than the speed of light, the proper time reduces to the absolute time, which is the time of classical mechanics. Our formulation can handle both cases. Nevertheless, the equations are derived in the framework of relativity in order to retain covariance for both space and time so that the form of the equations does not depend on a particular coordinate system i.e. that the form of the equations is the same independently of the coordinate transformations applied to them. As a result, symmetrical treatment of space and time is required in order to achieve invariant description.

Earlier on, we assumed that an electromagnetic field could be potentially present within and around the object. It is well known from electrodynamics [6] that the covariant Lagrangian density ${}_{*}\breve{L}$ associated with such a field can be written as

(25) $${}_{*}\breve{L} = \tfrac{1}{16\pi} \sum_{\mu\nu} F^{\mu\nu}(x) F_{\mu\nu}(x) = \tfrac{1}{16\pi} \sum_{\alpha\beta\mu\nu} g_{\mu\alpha} g_{\nu\beta} F^{\alpha\beta}(x) F^{\mu\nu}(x)$$

where the antisymmetric electromagnetic tensor is defined as

$$(26) \qquad F^{\mu\nu}(x) \equiv \begin{bmatrix} 0 & -E_1(x) & -E_2(x) & -E_3(x) \\ E_1(x) & 0 & -B_3(x) & B_2(x) \\ E_2(x) & B_3(x) & 0 & -B_1(x) \\ E_3(x) & -B_2(x) & B_1(x) & 0 \end{bmatrix}$$

for which $\mathbf{E}(x)$ and $\mathbf{B}(x)$ are the electric and magnetic field respectively. It should be noticed that the Lagrangians define by equations (24) and (25) are both function of a matter tensor ($\frac{dx^\mu}{d\tau}$ and $F^{\mu\nu}(x)$ respectively) and the metric as assumed earlier in equation (20). Therefore, we have for the Lagrangian of the object:

$$(27) \qquad \breve{L} = {}_{>}\breve{L} + {}_{*}\breve{L}$$

If one substitutes equations (24), (25) and (27) into equations (23) one obtains:

$$(28) \quad \breve{T}^{\mu\nu} = \tfrac{1}{2}\rho(x)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} + \tfrac{1}{4\pi}\left(\tfrac{1}{4}\sum_{\alpha\beta} g^{\mu\nu} F^{\alpha\beta}(x) F_{\alpha\beta}(x) - \sum_{\beta} F^{\mu\beta}(x) F^{\nu}_{\beta}(x)\right)$$

Finally, it one substitutes the value of the energy-momentum tensor density in equations (22) one obtains:

$$(29) \qquad R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu}\, R = \kappa\, \breve{T}_{\mu\nu}$$

which is a set of ten (because of the symmetry properties of the tensors) covariant non-linear equations describing the relations in between the energy-momentum content of the object and the curvatures of the associated Riemannian space. That is the relation we were looking for! Such a system of equations is very difficult to solve. In order to integrate equations (29), we assume that the Riemannian space is weakly curved [6] which means that

$$(30) \qquad g_{\mu\nu}(x) \approx \eta_{\mu\nu} + h_{\mu\nu}(x)$$

where

$$(31) \qquad \eta_{\mu\nu} \equiv \mathbf{diag}(1,-1,-1,-1) \text{ and } \left|h_{\mu\nu}(x)\right| << 1$$

The quasi-flatness condition can be achieved by a judicious choice of the constant κ, which determines the weight of the energy-momentum tensor i.e. the source term in equation (29). Nevertheless, the equations remain applicable to a very broad class of objects. If equations (30) and (31) are satisfied, one can neglect the second-order terms in equations (22) which simplify after some manipulations to

$$\sum_{\alpha\beta} g^{\alpha\beta}\left(\frac{\partial^2 g_{\alpha\beta}}{\partial x^{\mu}\partial x^{\nu}} - \frac{\partial^2 g_{\nu\beta}}{\partial x^{\mu}\partial x^{\alpha}} - \frac{\partial^2 g_{\mu\alpha}}{\partial x^{\nu}\partial x^{\beta}} + \frac{\partial^2 g_{\mu\nu}}{\partial x^{\alpha}\partial x^{\beta}}\right) = \kappa\, \breve{T}_{\mu\nu} \tag{32}$$

These equations do not completely determine the metric [4, 6, 7]. That was to be expected because our equations are covariant and do not depend on a particular coordinate system. Consequently, one can use the coordinate system freedom or gauge freedom [6] in order to further simplify these equations. If one chooses the relation:

$$\sum_{\mu\nu} g^{\mu\nu}\left(\frac{\partial g_{\rho\mu}}{\partial x^{\nu}} - \tfrac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x_{\rho}}\right) = 0 \tag{33}$$

as the gauge, it can be shown that equations (32) reduce to

$$\sum_{\alpha\beta} g^{\alpha\beta}\frac{\partial^2 g_{\mu\nu}}{\partial x^{\alpha}\partial x^{\beta}} = \kappa\, \breve{T}_{\mu\nu} \tag{34}$$

The solution to such an equation can be obtained by following a Green's function [6] approach:

$$g_{\mu\nu}(x) = -\tfrac{\kappa}{4\pi}\iiint_D \frac{\breve{T}_{\mu\nu}\left(x_0 - |\mathbf{x}-\mathbf{x}'|, \mathbf{x}'\right)}{|\mathbf{x}-\mathbf{x}'|}\, d^3x' \tag{35}$$

Equation (35) can be integrated with Monte Carlo techniques. Once the metric has been evaluated, it is easy to obtain the Ricci tensor and scalar from equations (29). Equations (35) show that the metric at a given point is representative of the object as a whole since the integration is performed over the whole domain *D* (matter and field) spans by the object.

5 Definition of an Invariant Statistical Representation

Up to this point, we have associated a Riemann space to an object and we have characterized the curvature of this space by calculating the Ricci tensor and scalar distributions. Now, in order to obtain an invariant description, we must construct some invariant quantities from the Ricci curvatures. If one applies a coordinate transformation to the Ricci scalar, one obtains with the help of equations (6) and (16):

(36) $$R' = \sum_{\mu\nu} g'^{\mu\nu} R'_{\mu\nu} = = \sum_{\mu\nu\rho\sigma} \left(\frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} \right) g^{\mu\nu} \left(\frac{\partial x^{\mu}}{\partial x'^{\rho}} \frac{\partial x^{\nu}}{\partial x'^{\sigma}} \right) R_{\mu\nu} = R$$

Equation (36) shows that the Ricci scalar is invariant under arbitrary coordinate transformations and as a result we define our first ensemble of invariant quantities $\Re_1(x)$ as

(37) $$\{ \Re_1(x) | \Re_1(x) \equiv R(x) \}$$

If one computes the inner product of two Ricci tensors one obtains:

(38) $$\sum_{\mu\nu} R'_{\mu\nu} R'^{\mu\nu} = \sum_{\mu\nu\rho\sigma} \left(\frac{\partial x^{\mu}}{\partial x'^{\rho}} \frac{\partial x^{\nu}}{\partial x'^{\sigma}} \right) R_{\mu\nu} \left(\frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} \right) R^{\mu\nu} = \sum_{\mu\nu} R_{\mu\nu} R^{\mu\nu}$$

which is again invariant under arbitrary coordinate transformations. Consequently, we define our second ensemble of invariant quantities $\Re_2(x)$ as

(39) $$\left\{ \Re_2(x) | \Re_2(x) \equiv \sum_{\mu\nu} R_{\mu\nu}(x) R^{\mu\nu}(x) \right\}$$

As a result, an invariant statistical representation of the object can be constructed. The distributions of the ensembles defined by equations (37) and (39) are described by two histograms. The first histogram characterizes the distribution of the Ricci scalars while the second histogram characterized the distribution of the inner products of the Ricci tensors. More precisely, the histograms are defined as

(40) $$h_k(i) \equiv \sum_{\left\{ x \;\middle|\; \left[\left(i\Delta_k - \frac{\Delta_k}{2} \right) \le \Re_k(x) < \left(i\Delta_k + \frac{\Delta_k}{2} \right) \right] \right\}} \Re_k(x)$$

where Δ_k is the bin-width of histogram *k*. It is important to notice that, because of our symmetrical treatment of space and time, equations (40) provide a unique, unified and coupled description of matter and motion. In order to describe the matter only, one must perform the calculations in a coordinate system in which the object is at rest.

6 Conclusions

We have introduced a new approach for invariant description of multidimensional objects under arbitrary coordinate transformations based on an associated Riemannian space and a new histogram. Such a description is suitable for indexation and invariant recognition, and has application in many domains, including CAT scanning, multi-spectral vision and deformable objects.

In this discussion, we have restricted ourselves to multi-dimensional objects for which the motion, the mass density and the electromagnetic field are known. Our approach can be generalized to almost any kind of object. One should simply modify the Lagrangian in equation (27) in order to include the required additional terms. The new energy-momentum tensor density can be calculated from equations (23). By substituting the new energy-momentum tensor in equations (22), one obtains a new set of non-linear equations from which the metric, the Ricci scalar and the Ricci tensor can be evaluated. The invariant quantities and the corresponding histograms are defined, as previously, by equations (37), (39) and (40) respectively.

References

1. Anlei, Dong and Bhanu, B.: A New Semi-supervised EM Algorithm for Image Retrieval, Proceedings IEEE Computer Society Conference on Computer Vision and Pattern Recognition, Vol. 2 (2003) 662-667
2. Jong, Myeon Jeong and Young, Shik Moon: Content-based Video Retrieval Algorithms using Spatio-temporal Information about Moving Objects, Journal of Korea Inf. Sci. Soc.: Software and Applications, Vol. 29 (2002) 631-644
3. Vranic, Dejan V. and Saupe, D.: Description of 3D-Shape using a Complex Function on the Sphere, Proceedings IEEE International Conference on Multimedia and Expo, Vol.1 (2002) 177-180
4. Einstein, Albert: Die Grundlage der allgemeinen Relativitätstheorie, Annalen der Physik, Vol. 49 (1916)
5. Einstein, Albert: Hamiltonsches Princip und allgemeine Relativitätstheorie, Sitzungsberichte der Preussischen Akad. d. Wissenschaften (1916)
6. Dirac, Paul Adrien Maurice, General Theory of Relativity, Princeton University Press, Princeton (1996)
7. Lovelock David and Rund, Hanno: Tensors, Differential Forms, and Variational Principles, Dover Publications, New York (1989)