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## DOCUMENTATION PAGE

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<b>SUMMARY</b> Dynamically positioned vessels must maintain their heading and position entirely by appropriate manipulation of the actuators that they are equipped with. Typically, these actuators are fixed and/or azimuthing thrusters and rudders, although unconventional actuators may also be present such as hawsers or winches. The controller must apportion the desired body forces in an appropriate manner to the various actuators; this task is known as thrust allocation. This report describes the theory and practical techniques required to perform optimal thruster allocation.					
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# **Thrust Allocation Techniques for Dynamically Positioned Vessels**

LM-2008-04

J. Millan

January, 2008



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# 1 Executive Summary

Dynamically positioned vessels must maintain their heading and position entirely by appropriate manipulation of the actuators that they are equipped with. Typically, these actuators are fixed and/or azimuthing thrusters and rudders, although unconventional actuators may also be present such as hawsers or winches. The controller must apportion the desired body forces in an appropriate manner to the various actuators; this task is known as thrust allocation. This report describes the theory and practical techniques required to perform optimal thruster allocation.





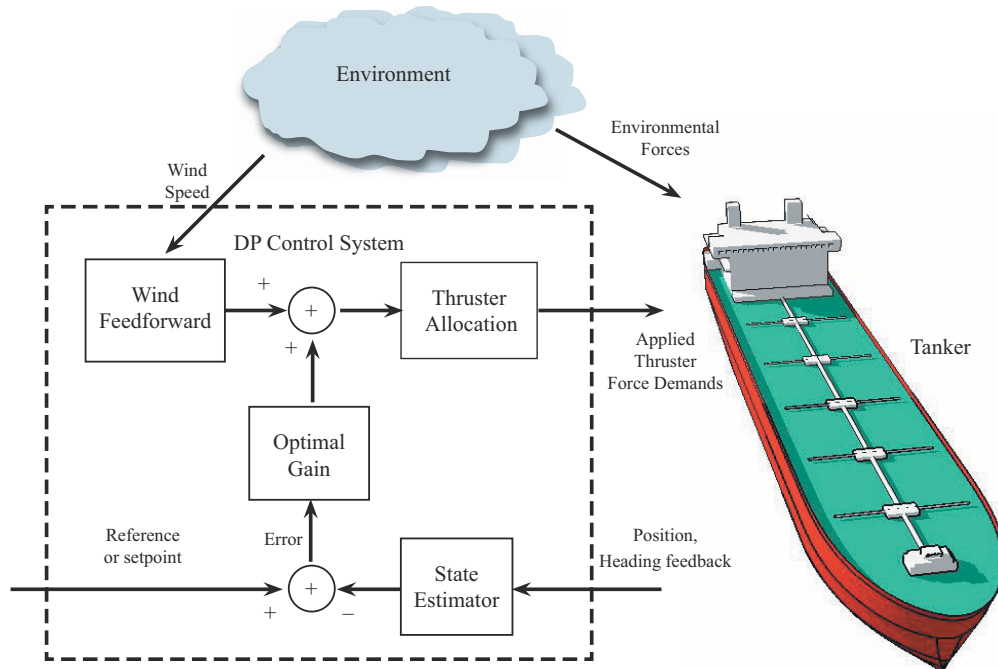


Figure 1: Block diagram of DP control system.

## 2 Dynamic Positioning Control

Figure 1 is a block diagram of a typical DP control system. The system is commanded with a 3 DOF setpoint command entered usually in earth referenced coordinates. The vessel's 3 DOF position  $x_e$  is measured with a variety of sensors and passed through a state estimator<sup>1</sup>. The error signal is computed in body coordinates, and control gains are applied to determine a controller demand. Measurements of the wind speed and direction are used to calculate a feedforward wind load, which is summed to the controller demand. A thruster allocation block determines how this controller  $\tau_c$  demand will be divided amongst the available thrusters, taking the geometry of their arrangement on the vessel into account. Not pictured in the figure is the optimal state estimation of current and wave generated forces and moment; these are also summed to produce the controller demand

<sup>1</sup>Typically, a Kalman filter is used to remove sensor noise.

vector. The controller demand vector is defined as follows:

$$\tau_c = [F_x \quad F_y \quad M_z]^T \quad (1)$$

where  $F_x$  and  $F_y$  are respectively, the controller demand forces in the surge and sway body axes and  $M_z$  is the demand moment in the yaw axis.

### 3 Thruster Allocation

The relation between the control demand and the individual actuator demand thrusts is as follows

$$\tau_c = T_a T_{th} \quad (2)$$

where  $T_{th}$  is a vector of thruster demands in Cartesian coordinates,  $\tau_c$  is the vector of thrust and moment demand from the controller (Eqn. 1), and  $T_a$  is the thruster allocation matrix, defined as follows:

$$T_{th} = [T_{1x} \quad T_{1y} \quad T_{2x} \quad T_{2y} \quad \dots \quad T_{nx} \quad T_{ny}]^T$$

where  $n$  is the number of thrusters, so that  $T_{th} \in \mathbb{R}^{2n}$ .

And the thruster arrangement matrix  $T_a \in \mathbb{R}^{3 \times 2n}$  is as follows:

$$T_a = \begin{bmatrix} 1 & 0 & 1 & 0 & \dots & 1 & 0 \\ 0 & 1 & 0 & 1 & \dots & 0 & 1 \\ l_{1y} & l_{1x} & l_{2y} & l_{2x} & \dots & l_{ny} & l_{nx} \end{bmatrix} \quad (3)$$

In equation 3, matrix entries of 1 indicate that 100% is available from the thruster if it is rotated to the appropriate direction. The bottom row are the lever arm distances that when combined with the thruster lateral and longitudinal forces, generate moment about the CG. Lever arm displacements are measured with respect to the *CG* of the vessel,  $l_{ix}$  and  $l_{iy}$  representing respectively, the longitudinal and lateral body displacements of the  $i$ th thruster. Each pair of columns in  $T_a$  corresponds to one azimuthing thruster: the odd numbered indexed columns correspond to the azimuthing thruster rotated to be aligned longitudinally with the vessel, and the even columns correspond to the thrusters rotated to be laterally aligned.

### 3.1 Solving the Thrust Equation

Owing to the fact the thruster arrangement matrix  $T_a$  is a so-called “fat” matrix, that is  $2n > 3$ , Eqn. 2 represents an underdetermined set of equations. Since there are more unknowns than there are equations there are many possible solutions that will satisfy Eqn. 2. One particular solution that is useful is the least-norm or minimum norm solution. The minimum norm solution of  $T_{th}$  requires finding the Moore-Penrose generalized inverse of  $T_a$  and premultiplying both sides of Eqn. 2:

$$T_{th} = T_a^\dagger \tau_c \quad (4)$$

where  $T_a^\dagger$  is the generalized inverse of  $T_a$ . The pseudo inverse is defined as follows:

$$T_a^\dagger = T_a^T (T_a T_a^T)^{-1} \quad (5)$$

The solution thrust vector  $T_{th}$  can be converted from Cartesian coordinates to an azimuth angle command and thrust demand pair  $[\alpha_i \ T_i]^T$  for each thruster as follows

$$T_{th} = [\alpha_1 \ T_1 \ \dots \ \alpha_n \ T_n]^T \quad (6)$$

$$\alpha_i = \arctan \frac{T_{iy}}{T_{ix}} \quad (7)$$

$$T_i = \sqrt{T_{ix}^2 + T_{iy}^2} \quad (8)$$

To avoid singularities,  $\arctan$  should be implemented as a typical  $\text{atan2}$  function:

$$\text{atan2}(y, x) = \begin{cases} \arctan\left(\frac{y}{x}\right) & \text{if } x > 0 \\ \pi + \arctan\left(\frac{y}{x}\right) & \text{if } y \geq 0, x < 0 \\ -\pi + \arctan\left(\frac{y}{x}\right) & \text{if } y < 0, x < 0 \\ \frac{\pi}{2} & \text{if } y > 0, x = 0 \\ -\frac{\pi}{2} & \text{if } y < 0, x = 0 \end{cases} \quad (9)$$

Note that  $T_i$  will always be positive. This corresponds to the thruster running in the positive thrust generation direction (i.e. propellor rotating one direction only) and thrust reversal is obtained by  $180^\circ$  rotation of the thruster. In light loading situations (i.e. very low sea state, low wind speed and low current speed) when the controller demands are small, the naive

conversion from cartesian to polar coordinates in Eqns. 7 and 8 may lead to very fast rotation of the azimuth thrusters. One possible solution to this problem can be to place an offset thrust on one or more thrusters for the others to “pull” against. Hysteresis may be employed, but it is generally an undesirable behaviour to include in a closed loop system. The rapid azimuthing issue may not be a problem if appropriate state estimation techniques have been used in the controller, leading to a smoothed, noise free demand signal.

### 3.2 Thruster Saturation

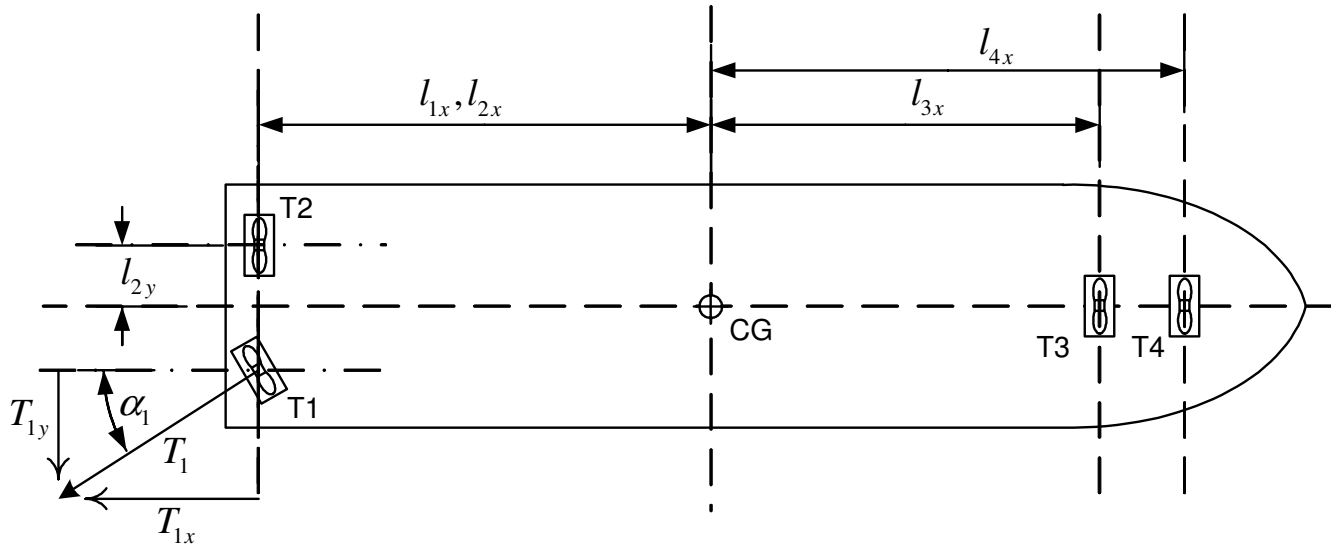
While the thrusts given using the pseudo-inverse technique are minimal in a least-squares sense, the solution thrust for any given thruster may exceed the limit for that particular actuator,  $T_i > T_{i,max}$ . In an actual thruster, the maximum obtainable thrust  $T_{i,max}$  is dependent on many factors, including the speed of the thruster through the water, direction of thrust, and the proximity and wake direction of other thrusters, to name a few. If the thrusts are arbitrarily clamped at their respective saturation limits, then clearly the solution of Eqn. 2 will no longer hold, and the desired  $\tau_c$  will not be achieved. The resultant forces  $F_x$ ,  $F_y$  and moment  $M_z$  will be somewhat unpredictable, as they are now the result of whatever the clamped thruster forces will achieve. In most cases, this is sufficient for normal use, unless there is extreme saturation. On the other hand, if extreme saturation occurs, then the control system is not useful anyway.

### 3.3 Examples

The simple allocation technique described in section 3.1 works well when little or no saturation is present *and* the thrusters are all azimuthing thrusters that can be freely rotated to any azimuth angle. The first example that will be examined looks at a vessel equipped with 4 fully azimuthing thrusters. The second example outlines how to deal with vessels equipped with rudders and fixed tunnel thrusters.

#### 3.3.1 Azimuthing Thruster Allocation

In Fig. 2 an example vessel thruster arrangement is pictured. The corresponding thruster arrangement matrix is as follows:



**Figure 2:** Schematic of thruster arrangement of a vessel equipped with 4 azimuthing thrusters.

$$T_a = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ l_{1y} & l_{1x} & l_{2y} & l_{2x} & l_{3y} & l_{3x} & l_{4y} & l_{4x} \end{bmatrix} \quad (10)$$

The following example MATLAB code sets up the thruster arrangement matrix for the example vessel and then finds the pseudoinverse with the `pinv` function:

```
%=====
% Thrust Allocation Example
%
% X is body surge direction (+ to bow)
% Y is body sway direction (+ to port)
%
%=====
% Stbd. aft thruster location
l1x = -0.47;
l1y = 0.1;
% Port aft thruster location
l2x = -0.47;
l2y = -0.1;
% Forward thruster, on centerline
l3x = 0.45;
l3y = 0.0;
% Second forward thruster on centerline
l4x = 0.47;
l4y = 0.0;
% Thruster arrangement matrix
Ta = [1 0 1 0 1 0 1 0;
      0 1 0 1 0 1 0 1;
      l1y l1x l2y l2x l3y l3x l4y l4x];
% Find its generalized pseudo-inverse
Tainv = pinv(Ta);%
The pseudo inverse of the thruster arrangement matrix only needs to be
computed once, and then can be reused.
```

Tainv =

```

0.2474 0 -0.5254
0.0006 0.2500 0.1130
0.2474 0 -0.5254
-0.0006 0.2500 -0.1130
0.2526 0 0.5141
0 0.2500 0
0.2527 0 0.5367
0 0.2500 0

```

Alternatively, the generalized inverse can be computed using Eqn. 5:

```
Tainv = Ta'*(Ta*Ta')\(-1) % Ta' is the transpose of Ta
```

The next step is to compute the thrusts for a particular demand vector:

```

% Set up the demand vector
tauc = [0.5; 0.5; 1.0];
% Compute the cartesian body coordinate thrusts
Tth = Tainv*tauc;

```

The resulting thrust vector will be:

```

Tth =

-0.4017
0.2383
-0.4017
0.0117
0.6404
0.1250
0.6630
0.1250

```

The thrust vector is in cartesian coordinates but can be readily con-



verted to polar (azimuth, thrust) commands in MATLAB:

```
[theta,r] = cart2pol(Tth(1:2:end,:),Tth(2:2:end,:));
```

Which produces the following results for thrust and azimuth angles:

theta =

2.6062

3.1124

0.1928

0.1863

r =

0.4670

0.4018

0.6524

0.6747

The `cart2pol` function of MATLAB is vectorized. In another programming environment, it can be replicated by nesting Eqns. 7 and 8 in a loop, and indexing the even and odd indices of  $T_{th}$  to compute the angle and magnitude commands for the azimuth thrusters.

### 3.4 Thrust Equation With Weighting

A variation of the thrust allocation technique of §3.1 is to add a cost factor which allows the designer to weight the contribution of each of the individ-

ual thrusters by axis. Defining the weighting matrix as follows:

$$W = \begin{bmatrix} w_{1x} & & & & & & & 0 \\ & w_{1y} & & & & & & \\ & & w_{2x} & & & & & \\ & & & w_{2y} & & & & \\ & & & & \dots & & & \\ & & & & & w_{nx} & & \\ 0 & & & & & & w_{ny} & \end{bmatrix} \quad (11)$$

where  $w_{ix}$  is the cost to use the  $i$ th thruster in the surge axis, and  $w_{iy}$  is the cost to use them in the sway axis. The higher the cost, the less thrust that will be assigned to the thruster in the selected axis when calculating the solution. The weighted generalized inverse is derived in Appendix A, and is given by

$$T_W^\dagger = W^{-1}T_a^T (T_a W^{-1}T_a^T)^{-1} \quad (12)$$

Note that if the cost factors on the diagonal are all equal (for example the identity matrix  $W = I$ ), then the equation reduces to the pseudo-inverse.

For some examples, see §4 for comparisons between weighted and unweighted allocations for the same demand vectors.

### 3.4.1 Allocation for Non-Azimuthing Thrusters

Dealing with a fixed axis thruster is very simple. For example, if thruster  $T3$  in Fig. 2 is a tunnel thruster, then it is restricted to only generating lateral force, and thus the thruster arrangement matrix would be altered as follows:

$$T_a = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ l_{1y} & l_{1x} & l_{2y} & l_{2x} & 0 & l_{3x} & l_{4y} & l_{4x} \end{bmatrix} \quad (13)$$

since thruster  $T3$  cannot generate surge force, the  $T_a(1,5) = 0$  and  $T_a(3,5) = 0$ .

If thrusters  $T1$  and  $T2$  are non-azimuthing main thrusters, we get:

$$T_a = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ l_{1y} & 0 & l_{2y} & 0 & 0 & l_{3x} & l_{4y} & l_{4x} \end{bmatrix} \quad (14)$$

since both thrusters  $T1$  and  $T2$  can only generate surge force.

### 3.4.2 Allocation With Rudders

Consider a vessel equipped with two non-azimuthing main thrusters, 2 rudders and one tunnel bow thruster. The approach is to perform a two-step allocation process. Firstly, allocate only the sway and moment demands:

$$\bar{T}_{th} = \bar{T}_a^{-1} \bar{\tau}_c \quad (15)$$

where

$$\bar{\tau}_c = [F_y \quad M_z]^T \quad (16)$$

$$\bar{T}_a = \begin{bmatrix} 1 & 1 \\ l_{1x} & l_{rx} \end{bmatrix} \quad (17)$$

$$\bar{T}_{th} = [T_{bow} \quad T_{stern}]^T \quad (18)$$

Note that the pseudo inverse is not necessary since the matrix is already square. The resulting solution  $\bar{T}_{th}$  is the required lateral thrusts that will satisfy both the sway  $F_y$  and the moment  $M_z$  demands.<sup>2</sup>

Since in this example, only one bow thruster exists,  $T_{bow}$  (the required lateral thrust at the bow) will be assigned entirely to the bow tunnel thruster. The stern lateral thrust  $T_{stern}$  becomes the stern sway demand, which in this case, must be satisfied by the side thrust generated by the rudders. Of course, in order for the rudder to generate side thrust (at forward zero speed), one or both of the main thrusters must be operated ahead. One approach is to use the port main thruster for surge thrust generation only, so its rudder angle is set to  $\alpha_p = 0$ . Then the starboard rudder angle  $\alpha_s$  is set at maximum angle  $\alpha_{max}$  either to port or starboard, depending on the sign of the stern demand thrust  $T_{stern}$  that is needed.

$$\alpha_s = \begin{cases} \alpha_{max} & \text{if } T_{stern} > 0, \\ -\alpha_{max} & \text{if } T_{stern} \leq 0. \end{cases} \quad (19)$$

With  $\alpha_s = \pm\alpha_{max}$ , then either a 1 or  $-1$  is entered in the new thrust arrangement matrix:

---

<sup>2</sup>We will ignore the possibility of saturation at this point, and assume no saturation.

$$\bar{T}_{th} = \bar{T}_a^{-1} \bar{\tau}_c$$

where

$$\bar{\tau}_c = [F_x \quad T_{stern}]^T$$

$$\bar{T}_a = \begin{bmatrix} 1 & 1 \\ 0 & \pm 1 \end{bmatrix}$$

$$\bar{T}_{th} = [T_{port} \quad F_{stbd}]^T$$

Where  $T_{port}$  is the port main thruster setting and  $F_{stbd}$  is the desired sway force that must be generated by the starboard rudder. To find the corresponding thrust to set on the starboard main thruster,

$$T_{stbd} = \left| \frac{F_{stbd}}{c_r} \right| \quad (20)$$

where  $c_r$  is a coefficient relating the transfer of the main thrust to the side thrust for the rudder's maximum angle. The absolute value must be used, because the rudder only generates side force if the main thruster is operated in ahead thrust.

This step solves for the combined stern sway demand and the forward thrust. Mismatches in port and starboard main thrusts (i.e.  $T_{port} \neq T_{stbd}$ ) will generate a parasitic moment. Thus the final step is to subtract  $T_{stbd}$  from  $T_{port}$  to get the "adjusted"  $T_{port}$ .

It should be emphasized that this is one technique, but many others can be utilized. One disadvantage with this technique is that the starboard rudder has to move between two extremes (full port and full starboard angle). When the moment demand and sway demand net close to zero, there may be rapid fluctuation in the rudder between the two extremes. One approach to mitigate this is for small stern lateral demands, the rudder angle could be reduced or even calculated to be proportional to the demand.

### 3.5 Heading Priority Allocation

An alternative to simply clamping the thrusts in the event of thruster saturation, is to deal with it by implementing a priority system for allocation. Typically in DP control, the most important mode of control is to maintain

the vessel's heading: this is based on the premise that the bow would be pointing into the prevailing weather in order to minimize the wave and wind forces acting on the vessel. Thus, if the vessel is unable to maintain station with the bow oriented to minimize the load, then it would certainly be unable to maintain the station for other more unfavourable heading angles.

### 3.5.1 Allocate Yaw Moment

To implement the heading priority scheme, the first step should be to allocate thrusts as in Eqn. 4, and the magnitudes of each demand thrust examined. If any thrusters are saturated, the allocation step of Eqn 4, with the demand vector modified for a heading priority control strategy:

$$\tau_c = [0 \quad 0 \quad M_z]$$

In this case, the surge and sway demands have been eliminated and only the moment is allocated. At this point,  $T_{th}$  should be examined again for thruster saturation. If thrusters are still saturated, there is no recourse except to clamp the thrusts and continue.

In the case that there are no saturated thrusters after allocating the moment, this means that there is some reserve thrust capacity left in each thruster, but not enough to allocate the entire demand. Since meeting the yaw demand has priority, the azimuth angles and thrust levels are now optimum for this task. The next step will allocate the thrust required to satisfy the surge and sway, but this time with the azimuth angles fixed. Note that in all likelihood, the azimuth angles required for moment generation will be more conducive to sway force generation rather than surge.

### 3.5.2 Median Search Approach

The basic procedure will be to attempt to allocate some percentage of the surge and sway demand *given the fixed azimuth angles for moment*.

$$\begin{bmatrix} F_x \\ F_y \end{bmatrix} p_{ss} = \begin{bmatrix} \cos(\alpha_1) & \cos(\alpha_2) & \dots & \cos(\alpha_n) \\ \sin(\alpha_1) & \sin(\alpha_2) & \dots & \sin(\alpha_n) \end{bmatrix} \begin{bmatrix} \hat{T}_1 \\ \hat{T}_2 \\ \vdots \\ \hat{T}_n \end{bmatrix} \quad (21)$$

where  $\hat{T}_i$  are the component of thrust for each thruster that will satisfy the remaining surge-sway demand, given the existing azimuth angles for heading priority  $\alpha_i$ . The thrusts  $\hat{T}_i$  of Eqn. 21 are unknown and must be solved for (as before). This is accomplished by taking the pseudo-inverse of the new  $T_a$  matrix of Eqn. 21, (which is due to treating the azimuthing thrusters as fixed angle thrusters) and premultiplying both sides. The variable  $0 \leq p_{ss} \leq 1$  is the percentage of surge-sway demand to be allocated.

The thrusts computed here will represent the *additional* incremental thrust required to achieve  $p_{ss} \times 100\%$  of the sway and surge demand. Next, sum the thrusts required to achieve the moment (from the previous step) with this partial surge-sway thrust vector. If any thruster has saturated, reduce the surge-sway demand by 50% and try again. If no thrusters have saturated, increase the demand by 50% and try again. This process can be repeated iteratively until the thrusters are not saturated to within some threshold percentage match. For example, in 5 steps, the thruster capacity for this configuration will be utilized to within 3.125%

### 3.5.3 Additional Priority Levels

Instead of using  $p_{ss}$  the same for both  $F_x$  and  $F_y$ , a ratio could be chosen between surge and sway, which would reflect the relative importance of each with respect to the other.

## 4 Weighted Thrust Allocation

This section illustrates examples of weighted versus unweighted thruster allocation. This example will use the thruster arrangement of Fig. 2, and assumes that all thrusters are azimuthing thrusters. Recall that for neutral weighting, the identity matrix  $W = I$  and the thrust allocation is equivalent to the pseudo-inverse.

### 4.1 Accounting for Mismatched Thrusters

If each azimuthing thruster has the same rating, then it would make sense have the same surge and sway weighting factors for each thruster. In this case, assume that the bow thrusters 3, 4 have a lower rating than those at the stern, thrusters 1, 2. In order to encourage the higher-rated thrusters

at the stern, modify the weighting matrix, so that the diagonal elements  $w_{11}, w_{22}, w_{33}, w_{44} = 0.25$  have a lower cost; this causes the thrust solution to rely more heavily on the stern thrusters.

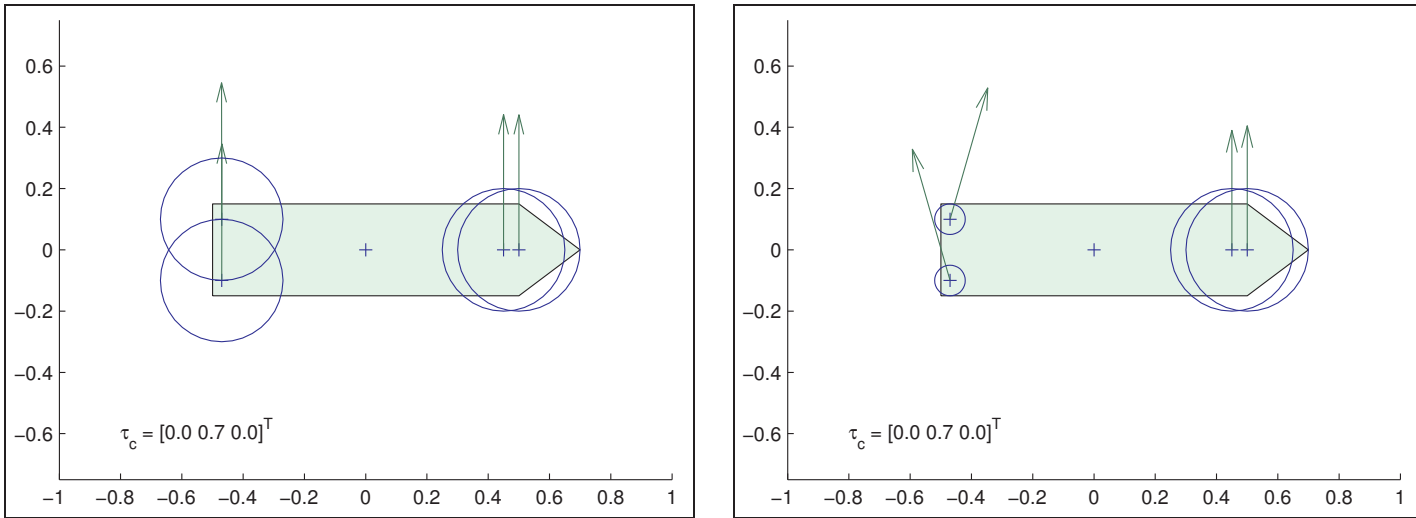
$$W = \begin{bmatrix} 0.25 & & & & & & & 0 \\ & 0.25 & & & & & & \\ & & 0.25 & & & & & \\ & & & 0.25 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ 0 & & & & & & & 1 \end{bmatrix} \quad (22)$$

The results of the allocation using a straightforward pseudo-inverse and the weighting matrix given above are given in Fig. 3. In the figure, the cost for each thruster is graphically portrayed by the blue circles. The effect of lowering the cost on the stern thrusts is a small decrease in the bow thrusts accompanied by a moderate increase in the stern thrusts. The demand from the controller is a pure sway force only, thus in order to balance the increased emphasis on the stern thrusters, they must be rotated to cancel the moment generated by the decrease in overall thrust near the bow.

## 4.2 Preventing Thruster Interaction

Due to the arrangement of the thrusters on the target vessel, the main thrusters will interact with each other when aligned laterally across the vessel. This is particularly evident Fig. 3 (left) in which the two stern thrusters have aligned. In this case one thruster will direct it's wash directly at the other one. A similar situation will occur in the bow thrusters when they must align to generate surge. Assigning differing costs in surge and sway axes for each azimuthing thruster can effectively discourage allocation solutions that lead to these situations.

To prevent thruster interaction in the stern thrusters 1, 2, the weighting matrix has been modified to increase the cost factor to the sway axis as



**Figure 3:** Comparison of allocations:(left) straightforward pseudo-inverse; (right) main thrusters have higher capacity and have been given a lower cost factor. The blue circle graphically represents the costs assigned to each thrusters.



follows:

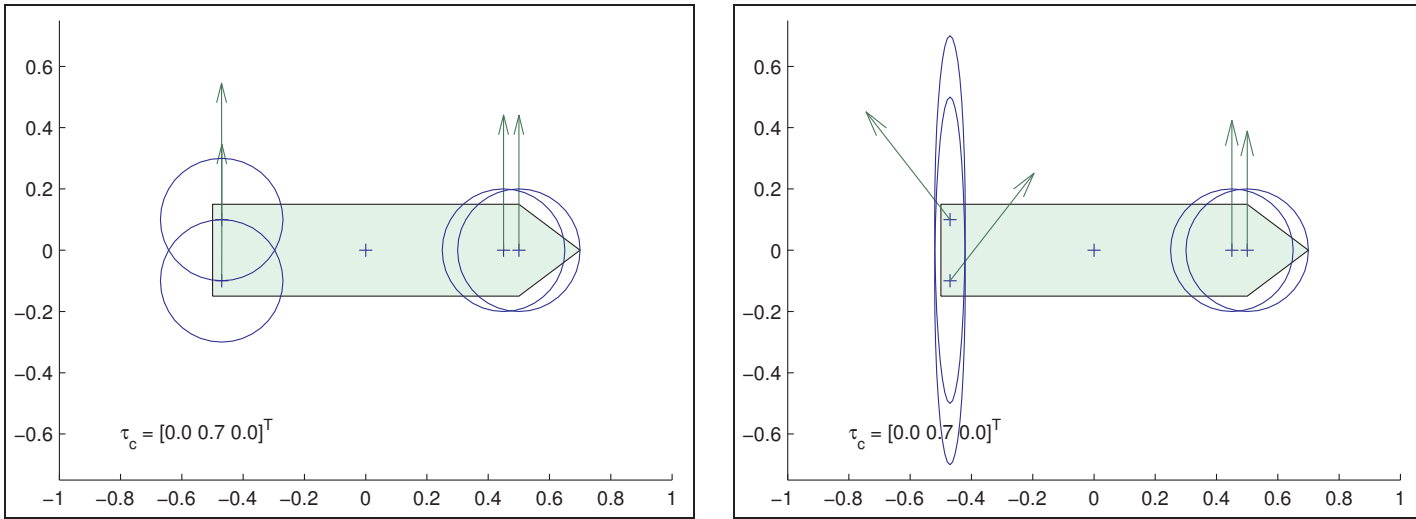
$$W = \begin{bmatrix} 0.25 & & & & & & 0 \\ & 3 & & & & & \\ & & 0.25 & & & & \\ & & & 3 & & & \\ & & & & 1 & & \\ & & & & & 1 & \\ & & & & & & 1 \\ 0 & & & & & & & 1 \end{bmatrix} \quad (23)$$

The result, illustrated in Fig. 4 (right), is that the stern thrusters are even more significantly displaced in azimuth, thus avoiding the situation in which either of the thrusters direct their propellor wash at each other. The unweighted example is repeated on the left for comparison purposes.

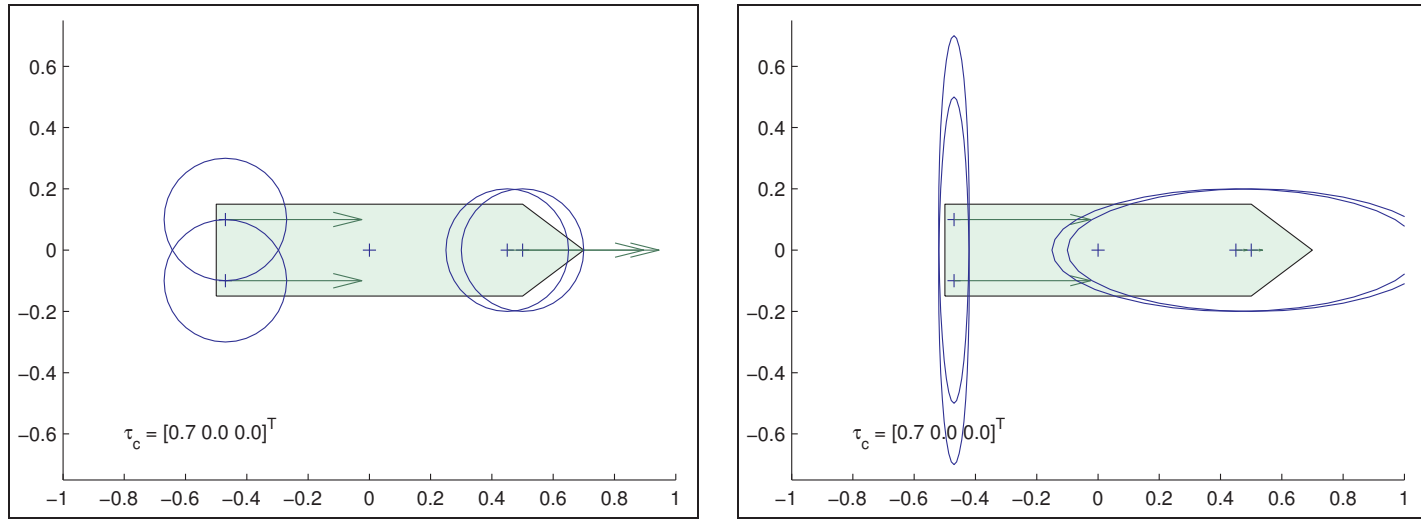
In the next example, use of pure surge thrust by the bow thrusters is discouraged by increasing the costs for these thrusters in the surge axis, as follows:

$$W = \begin{bmatrix} 0.25 & & & & & & 0 \\ & 3 & & & & & \\ & & 0.25 & & & & \\ & & & 3 & & & \\ & & & & 3 & & \\ & & & & & 1 & \\ & & & & & & 3 \\ 0 & & & & & & & 1 \end{bmatrix} \quad (24)$$

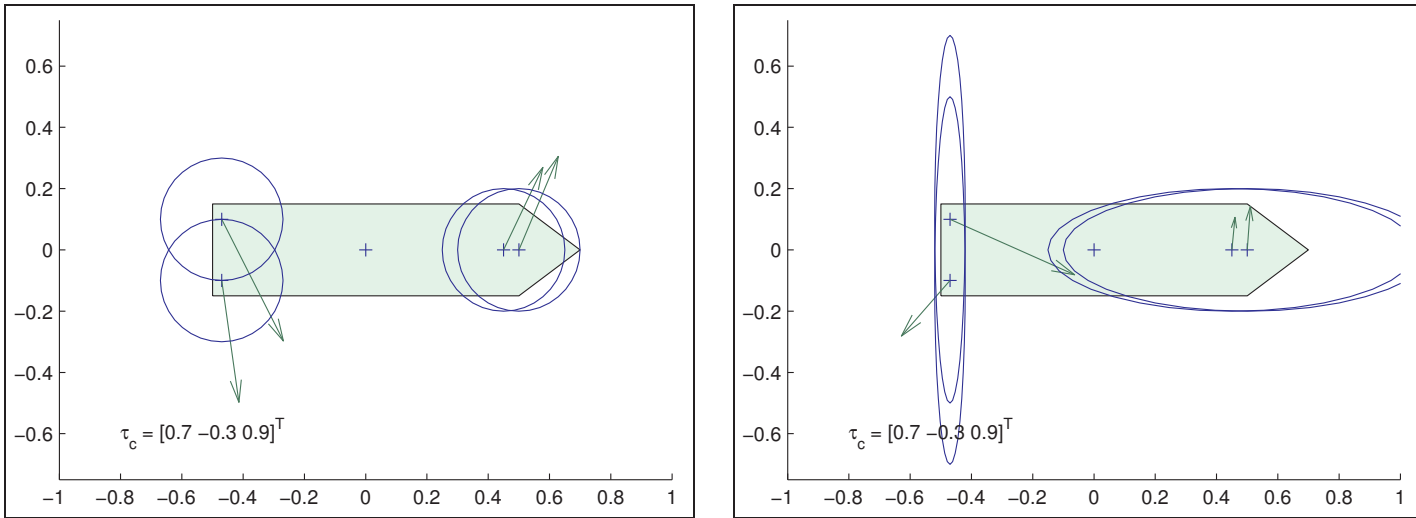
The next figures present comparisons between unweighted solutions versus the effect of the weighting matrix of Eqn. 24 (above) on the thrust vector solution for a pure surge demand (Fig. 5), and a random demand mixed demand (Fig. 6).



**Figure 4:** Comparison of allocations for sway demand only:(left) straightforward pseudo-inverse; (right) main thrusters have been given a higher cost factor in sway.



**Figure 5:** Comparison of allocations for pure surge demand:(left) pseudo inverse; (right) weighting matrix of Eqn. 24.



**Figure 6:** Comparison of allocations for a randomly chosen demand  $\tau_c = [0.7, -0.3, 0.9]^T$  :(left) pseudo-inverse; (right) weighting matrix of Eqn. 24.



## References

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# Appendices





## A Derivation of the Weighted Pseudo-Inverse

Consider the quadratic energy cost function

$$J = \frac{1}{2}u^T W u \quad (25)$$

where  $u \in \mathbb{R}^{2 \times n}$  is the vector of thrusts ( $T_{th}$ ) for  $n$  actuators in cartesian body coordinates,  $W \in \mathbb{R}^{n \times n}$  is the positive definite weighting matrix, and  $J$  is a scalar which is to be minimized, subject to the constraint that the candidate solutions  $u$  must equal the demanded thrust  $\tau_c \in \mathbb{R}^3$ :

$$\tau_c - T_a u = 0 \quad (26)$$

The technique of Lagrange multipliers will be used to solve this constrained optimization problem. A constrained optimization problem can be posed as follows: given some differentiable function  $f(x) : \mathbb{R}^n \rightarrow \mathbb{R}$  that is to be minimized (or maximized), combined with the restriction that the solution  $x$  satisfies some constraint of equality such that  $g(x) = 0$ .

The *Lagrangian* is defined as

$$\Lambda(x, \lambda) = f(x) - \lambda g(x) \quad (27)$$

where  $\lambda$  is a vector of Lagrange multipliers. Optimal points occur where the gradient of the Lagrangian is zero:

$$\nabla \Lambda(x, \lambda) = 0 \quad (28)$$

We wish to minimize Eqn. 25 subject to Eqn. 26, so the Lagrangian for the weighted thrust allocation is

$$\Lambda(u, \lambda) = \frac{1}{2}u^T W u - \lambda (\tau_c - T_a u) \quad (29)$$

while looking for solutions that satisfy

$$\nabla \Lambda(u, \lambda) = \nabla \left( \frac{1}{2}u^T W u \right) - \lambda \nabla (\tau_c - T_a u)$$

and,

$$\nabla \Lambda(u, \lambda) = 0$$

Differentiating the Lagrangian  $\Lambda$  with respect to  $u$  yields the following expression:

$$\nabla\Lambda(u, \lambda) = Wu - T_a^T \lambda \quad (30)$$

Solving Eqn. 30 for  $u$  yields:

$$u = W^{-1}T_a^T \lambda \quad (31)$$

From the constraint equation Eqn 26 it is known that

$$\tau_c = T_a u \quad (32)$$

Substituting 31 into 32:

$$\tau_c = T_a W^{-1} T_a^T \lambda$$

Providing that  $(T_a W^{-1} T_a^T)$  is non-singular, the optimal solution for the Lagrange multipliers is as follows:

$$\lambda = (T_a W^{-1} T_a^T)^{-1} \tau_c \quad (33)$$

Substituting this result into Eqn. 31 yields the optimal  $u$ :

$$u = W^{-1} T_a^T (T_a W^{-1} T_a^T)^{-1} \tau_c \quad (34)$$

thus the weighted pseudo-inverse of matrix  $T_a$  is

$$\boxed{T_a^\dagger = W^{-1} T_a^T (T_a W^{-1} T_a^T)^{-1}} \quad (35)$$

## A.1 Unweighted Pseudo Inverse

The pseudo-inverse (i.e. unweighted) is simply a special case of the weighted pseudo inverse. Redefining the cost function as follows:

$$J = \frac{1}{2} u^T u \quad (36)$$

This is equivalent to equation 25 with  $W$  equal to the identity matrix. Substituting  $I$  into 34 results in

$$u = \underbrace{T_a^T (T_a T_a^T)^{-1}}_{T_a^\dagger} \tau_c \quad (37)$$

is the minimum energy solution such that the constraint of 32 is satisfied. Thus, the pseudo-inverse or generalized inverse of  $T_a$  is

$$\boxed{T_a^\dagger = T_a^T (T_a T_a^T)^{-1}} \quad (38)$$