Supplemental Document

OPTICA QUANTUM

Correlations for subsets of particles in symmetric states: what photons are doing within a beam of light when the rest are ignored: supplement

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Supplement for "Correlations for symmetric states: what subsets of photons are doing within a beam of light"

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1. Supplement 1

We here directly prove that

$$\hat{\varrho}(q|\hat{\rho}) \equiv \sum_N P(q\,\text{from}\,N)\hat{\varrho}(q|\hat{\rho}_N) \propto \sum_{|\mathbf{m}|=|\mathbf{n}|=q} \frac{\langle \hat{O}_{\mathbf{mn}}\rangle_{\hat{\rho}}}{\sqrt{m_1!\cdots m_d! n_1!\cdots n_d!}} |\mathbf{n}\rangle\langle\mathbf{m}|. \tag{1}$$

We begin with our arbitrary state expressed in the Fock basis as $\hat{\rho} = \sum_{\mathbf{kj}} \hat{\rho}_{\mathbf{kj}} |\mathbf{k}\rangle\langle\mathbf{j}|$. Using the projectors onto the N-photon subspaces,

$$\hat{P}_N = \sum_{|\mathbf{k}|=N} |\mathbf{k}\rangle\langle\mathbf{k}|, \tag{2}$$

we find the N-photon components of the state to be

$$p_N\hat{\rho}_N \equiv \hat{P}_N\hat{\rho}\hat{P}_N = \sum_{|\mathbf{k}|=|\mathbf{j}|=N} \hat{\rho}_{\mathbf{kj}} |\mathbf{k}\rangle\langle\mathbf{j}|. \tag{3}$$

Note that we set the states $\hat{\rho}_N$ to be normalized to unity, with p_N the corresponding probability of the state actually having N photons. To trace out $N-q$ photons, we apply

$$\begin{aligned} \mathrm{Tr}_{N-q}(p_N\hat{\rho}_N) &= p_N\frac{q!}{N!}\sum_{i_1,\cdots,i_{N-q}} \hat{a}_{i_1}\cdots\hat{a}_{i_{N-q}}\hat{\rho}_N\hat{a}^\dagger_{i_{N-q}}\cdots\hat{a}^\dagger_{i_1} \\ &= p_N\frac{q!}{N!}\sum_{|\mathbf{l}|=N-q}\binom{N-q}{\mathbf{l}}\hat{a}_1^{l_1}\cdots\hat{a}_d^{l_d}\hat{\rho}_N\hat{a}_1^{\dagger l_1}\cdots\hat{a}_d^{\dagger l_d}, \end{aligned} \tag{4}$$

where we use the multinomial coefficient because the creation operators for each mode commute with each other. Now the action on a term $|\mathbf{k}\rangle\langle\mathbf{j}|$ requires $k_i - l_i \geq 0$ and $j_i - l_i \geq 0$ for all modes i. We find the above to equal

$$\begin{aligned} \mathrm{Tr}_{N-q}(p_N\hat{\rho}_N) &= \frac{q!}{N!}\sum_{|\mathbf{l}|=N-q}\sum_{|\mathbf{k}|=|\mathbf{j}|=N}\binom{N-q}{\mathbf{l}}\sqrt{\frac{k_1!\cdots k_d! j_1!\cdots j_d!}{(k_1-l_1)!\cdots(k_d-l_d)!(j_1-l_1)!\cdots(j_d-l_d)!}} \\ &\quad\times\hat{\rho}_{\mathbf{kj}}|\mathbf{k}-\mathbf{l}\rangle\langle\mathbf{j}-\mathbf{l}| \\ &= \binom{N}{q}^{-1}\sum_{|\mathbf{m}|=|\mathbf{n}|=q}\sum_{|\mathbf{k}|=|\mathbf{j}|=N}\sqrt{\binom{k_1}{m_1}\cdots\binom{k_d}{m_d}\binom{j_1}{n_1}\cdots\binom{j_d}{n_d}}\hat{\rho}_{\mathbf{k},\mathbf{j}}|\mathbf{m}\rangle\langle\mathbf{n}|\delta_{\mathbf{k}-\mathbf{m},\mathbf{j}-\mathbf{n}}. \end{aligned} \tag{5}$$

The binomial coefficients restrict the sums to the appropriate domains.

Next, we identify $P(q \,\text{from}\, N) = p_N\binom{N}{q}/\mathcal{N}(q)$ as the probability of q photons originating from the N-photon sector and $\hat{\varrho}(q|\hat{\rho}_N) = \text{Tr}_{N-q}(\hat{\rho}_N)$ as the state from which they must have originated. Then we can combine our calculations for the entire state to yield

$$\begin{aligned}
\hat{\varrho}(q|\hat{\rho}) &= \frac{1}{\mathcal{N}(q)} \sum_N \sum_{|\mathbf{m}|=|\mathbf{n}|=q} \sum_{|\mathbf{k}|=|\mathbf{j}|=N} \sqrt{\binom{k_1}{m_1}\cdots\binom{k_d}{m_d}\binom{j_1}{n_1}\cdots\binom{j_d}{n_d}} \hat{\rho}_{\mathbf{k},\mathbf{j}} \left|\mathbf{m}\right\rangle \left\langle\mathbf{n}\right| \delta_{\mathbf{k}-\mathbf{m},\mathbf{j}-\mathbf{n}} \\
&= \frac{1}{\mathcal{N}(q)} \sum_{|\mathbf{m}|=|\mathbf{n}|=q} \frac{\langle \hat{O}_{\mathbf{nm}} \rangle_{\hat{\rho}}}{\sqrt{m_1!\cdots m_d! n_1!\cdots n_d!}} \left|\mathbf{m}\right\rangle \left\langle\mathbf{n}\right| .
\end{aligned} \tag{6}$$

Swapping the index labels $\mathbf{m} \leftrightarrow \mathbf{n}$ reproduces the result in the main text.

2. Supplement 2

We next explain the connection between loss channels as usually discussed in the literature and our formulation. Typically, a loss channel for mode i is described by enacting the input-output relation

$$\hat{a}_i \to \sqrt{\eta_i}\hat{a}_i + \sqrt{1-\eta_i}\hat{b}_i \tag{7}$$

and then tracing out the bosonic mode annihilated by $\hat{b}_i$ that began in its vacuum state. One can notice by these input-output relations that, if every channel has the same transmission parameter $\eta_i = \eta$, then the a linear optical transformation $\hat{a}_i \to \sum_{ij} U_j \hat{a}_j$ commutes with the overall loss channels because their actions simply differ by a unitary transformation among the vacuum modes. It is striking to compare this with our operation Tr_1, which also commutes with the action of linear optical networks and thus has the same effect whether it acts before or after such a network. Surely these two phenomena must be connected, and indeed they are.

A loss channel enacts a beam splitter between modes $\hat{a}_i$ and $\hat{b}_i$. Its action is equivalent to acting on a global state supplanted with an auxiliary vacuum mode that gets traced out:

$$\hat{\rho} \underset{\eta_i}{\to} \sum_n {}_{b_i}\langle n|\hat{B}(\eta_i)(\hat{\rho} \otimes |0\rangle_{b_i} \langle 0|)\hat{B}^\dagger(\eta_i)\,|n\rangle_{b_i} . \tag{8}$$

The beam-splitter operation can take many equivalent forms, the most useful of which for our purposes is

$$\begin{aligned}
\hat{B}(\eta_i) &= \exp\Big(\sqrt{\frac{1-\eta_i}{\eta_i}}\hat{a}_i\hat{b}_i^\dagger\Big) \exp\big((\hat{a}_i^\dagger\hat{a}_i - \hat{b}_i^\dagger\hat{b}_i)\ln\sqrt{\eta_i}\big) \exp\Big(-\sqrt{\frac{1-\eta_i}{\eta_i}}\hat{a}_i^\dagger\hat{b}_i\Big) \\
\Rightarrow \quad {}_{b_i}\langle n|\,\hat{B}(\eta_i)\,|0\rangle_{b_i} &= {}_{b_i}\langle n|\exp\Big(\sqrt{\frac{1-\eta_i}{\eta_i}}\hat{a}_i\hat{b}_i^\dagger\Big)\,|0\rangle_{b_i} \exp(\hat{a}_i^\dagger\hat{a}_i\ln\sqrt{\eta_i}) \\
&= \sum_n \frac{1}{\sqrt{n!}}\left(\frac{1-\eta_i}{\eta_i}\right)^{n/2} \hat{a}_i^n \sqrt{\eta_i}^{\hat{a}_i^\dagger\hat{a}_i},
\end{aligned} \tag{9}$$

which can be verified for its action $\hat{B}(\eta_i)\hat{a}_i\hat{B}^\dagger(\eta_i) = \sqrt{\eta_i}\hat{a}_i + \sqrt{1-\eta_i}\hat{b}_i$. When each mode has the same loss channel, the state evolves as

$$\begin{aligned}
\hat{\rho} &\underset{\eta}{\to} \sum_{n_1,\cdots,n_d} \frac{1}{n_1!\cdots n_d!}\left(\frac{1-\eta}{\eta}\right)^{n_1+\cdots+n_d} \hat{a}_1^{n_1}\sqrt{\eta}^{\hat{a}_1^\dagger\hat{a}_1}\cdots\hat{a}_d^{n_d}\sqrt{\eta}^{\hat{a}_d^\dagger\hat{a}_d}\hat{\rho}\sqrt{\eta}^{\hat{a}_d^\dagger\hat{a}_d}\hat{a}_d^{\dagger n_d}\cdots\sqrt{\eta}^{\hat{a}_1^\dagger\hat{a}_1}\hat{a}_1^{\dagger n_1} \\
&= \sum_N \frac{\left(\frac{1-\eta}{\eta}\right)^N}{N!} \sum_{|\mathbf{n}|=N} \binom{N}{\mathbf{n}} \hat{a}_1^{n_1}\cdots\hat{a}_d^{n_d}\sqrt{\eta}^{\hat{N}}\hat{\rho}\sqrt{\eta}^{\hat{N}}\hat{a}_d^{\dagger n_d}\cdots\hat{a}_1^{\dagger n_1}.
\end{aligned} \tag{10}$$

The multinomial coefficient tells us how many times each distribution of modes operators should be counted; this is equivalent to having N different operators that can each come from one of d different modes. By using the alternate counting, we find

$$\begin{aligned}\hat{\rho} \underset{\eta}{\rightarrow} &\sum_N \frac{\left(\frac{1-\eta}{\eta}\right)^N}{N!} \sum_{i_1=1}^{d} \cdots \sum_{i_N=1}^{d} \hat{a}_{i_1} \cdots \hat{a}_{i_N} \sqrt{\eta}^{\hat{N}} \hat{\rho} \sqrt{\eta}^{\hat{N}} \hat{a}_{i_N}^{\dagger} \cdots \hat{a}_{i_1}^{\dagger} \\ &= \sum_N \frac{\left(\frac{1-\eta}{\eta}\right)^N}{N!} \underbrace{\mathrm{Tr}_1(\sqrt{\hat{N}} \cdots}_{N-1 \text{ more times}} \mathrm{Tr}_1(\sqrt{\hat{N}} \sqrt{\eta}^{\hat{N}} \hat{\rho} \sqrt{\eta}^{\hat{N}} \sqrt{\hat{N}}) \cdots \sqrt{\hat{N}}).\end{aligned} \tag{11}$$

The continuous loss channel can thus be expressed in terms of convex combinations tracing out individual photons N times, but with a photon-number-dependent factor being applied before each individual photon is traced out. Each component of this expression is independent from mode decomposition and thus commutes with the action of linear optical transformations on $\hat{\rho}$.

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