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Numerical Evaluation of Marine Riser Dynamics

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SUMMARY

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This project involves high mode vibration tests on an instrumental riser in uniform flow and in sheared flow; particle image velocimetry of wakes behind large riser models; pushing the bounds of Computational Fluid Dynamics behind a moving cylinder; numerical prediction of riser vibration in the time domain; and fatigue work. Dr. Bose and his team will develop model tests on instrumented risers at IOT, scale tests in the natural test site at Bonne Bay, numerical simulations, and both structural and hydrodynamic and fatigue studies of riser materials and design [1].

This report documents my role in this project in helping to develop the numerical prediction program created by Dr. Wayne Raman-Nair of IOT. My role was to test the program against known results to validate its ability to simulate riser dynamics. I also investigated the effect of lift coefficient on riser vibration amplitude and simulated a 3000m riser to see how the lift coefficient effected the tension and bending moment.

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TABLE OF CONTENTS

ABSTRACT.....i
TABLE OF CONTENTS..... ii
LIST OF FIGURES..... ii
LIST OF TABLES..... iii

1.0 INTRODUCTION..... 1
2.0 BACKGROUND.....1
2.1 Glossary of Parameters..... 2
3.0 TESTING AGAINST KNOWN RESULTS.....2
3.1 Proofs.....2
3.1.1 Hanging Chain.....2
3.1.2 Catenary Profile.....4
3.1.3 Harmonic Loading on a Solid Beam..... 5
3.2 Experimental Results.....7
4.0 LIFT EXPERIMENT.....9
5.0 LONG RISER EXPERIMENT..... 11
5.1 Lift Coefficient Results..... 12
5.1.1 Lift Coefficient of 0.2..... 12
5.1.2 Lift Coefficient of 0.4..... 13
5.1.3 Lift Coefficient of 0.6..... 14
5.1.4 Lift Coefficient of 0.8..... 15
5.1.5 Lift Coefficient of 1.0..... 16
5.2 Parameter Relationships..... 17
6.0 CONCLUSION AND RECOMMENDATIONS 19
7.0 REFERENCES..... 20

LIST OF FIGURES

Figure 1: Von Karman Street..... 1
Figure 2: Schematic of lumped masses.....3
Figure 3: Hanging chain vs. riser natural frequency..... 3
Figure 4: Catenary Profile..... 4
Figure 5: Catenary Testing Results..... 5
Figure 6: Harmonic loading on a simply supported beam.....5
Figure 7: Zeta's effects on deflection.....6
Figure 8: Trim riser..... 7
Figure 9: CF vs. IL amplitude.....8
Figure 10: CF vs. IL frequency.....9
Figure 11: Bonne Bay setup.....9
Figure 12: Lift vs. Amplitude.....10
Figure 13: Long Riser setup..... 11
Figure 14: Maximum tension for lift 0.2..... 12
Figure 15: Maximum bending moment for lift 0.2..... 12
Figure 16: Maximum tension for lift 0.4..... 13
Figure 17: Maximum bending moment for lift 0.4..... 13

Figure 18: Maximum tension for lift 0.6 14
Figure 19: Maximum bending moment for lift 0.6 14
Figure 20: Maximum tension for lift 0.8 15
Figure 21: Maximum bending moment for lift 0.8 15
Figure 22: Maximum tension for lift 1.0 16
Figure 23: Maximum bending moment for lift 1.0 16
Figure 24: Maximum tension vs. lift 17
Figure 25: Maximum tension amplitude vs. lift 17
Figure 26: Maximum bending moment vs. lift 18
Figure 27: Maximum bending moment amplitude vs. lift 18

LIST OF TABLES

Table 1: Natural frequencies of a hanging chain and a riser 4
Table 2: Test amplitudes for different zetas 6
Table 3: Changing variables to simulate real experiment 7
Table 4: Maximum Deflections of Each Lumped Mass 8
Table 5 Effect of increasing lift on amplitude 10

APPENDICES

APPENDIX A: Proofs

NUMERICAL INVESTIGATION OF MARINE RISER DYNAMICS

1.0 INTRODUCTION

Vortex induced vibration (VIV) is a very prominent research topic today and has many applications in the engineering field. It is of particular importance in offshore oil production systems as it can cause structural and environmental damage to marine risers. As oil companies are forced to search in deeper waters to find oil reserves, the length of the riser increases and with this the impact of VIV.

Vortices are shed from either side of marine risers in what is called a von Karman Vortex Street and generally occur at a regular frequency called the shedding frequency (n_s). When the shedding frequency is close to the natural frequency of the riser (n_b), there is a resonant condition where significant vibration occurs [2]. This is termed vortex induced vibration.

It is quite costly and often not feasible to perform experimental studies on marine risers, so researchers are turning to numerical simulation to calculate the vibrational effects [3]. Many programs have been created to simulate VIV, most use a modal analysis method where the measured vibration is separated into different frequency modes. There is considerably error between predictions of marine riser VIV fatigue damage by computer models and observed damage, often by orders of magnitude [4].

Figure 1 Von Karman Vortex Street.

2.0 BACKGROUND

Dr. Wayne Raman-Nair of IOT has created a program that simulates riser dynamics and is based on a time-domain analysis. The riser is simulated using lumped masses connected by springs that model the riser's properties, such as its bending and extensional stiffness. The number of lumped masses can be changed, depending on how accurate you want the results. They may, however, be reduced in favour of a shorter run time. Many parameters have been taken in to consideration in the program, such as constant or sheared fluid flow, length, inner/outer riser diameter, internal flow rate etc. This allows many specific situations to be modelled.

2.1 Glossary of Parameters

These are some parameters from the marine riser code that are used throughout the report.

A: cross sectional area [m^2]
CL0: lift coefficient
E: Modulus of Elasticity [N/m^2]
F0: force per unit length [N/m]
g: gravity [9.81 m/s^2]
I: Moment of Inertia [m^4]
L0: natural length of the riser [m]
n: number of lumped masses
omega_beam: forcing frequency (w) [rad/s]
rhof: fluid density [kg/m^3]
rho_material: 'riser' density [kg/m^3]
sqrt(n): square root of n
tspan[0:0.25:10]: time span of the test run is from 0 to 10 seconds in intervals of 0.25 seconds
vfluid: velocity profile of current flow [m/s]
v(x): Deflection at point x [m]
x: distance along beam [m]
zeta: structural damping coefficient

3.0 TESTING AGAINST KNOWN RESULTS

My main role in this project was to test Dr. Raman-Nair's marine riser program against known results to develop its validity in simulating riser dynamics. Two sources for known results are proofs and experimental data. I compared the program against three proofs: the hanging chain, the catenary profile, and the harmonic loading of a horizontal beam. Known results were more difficult to locate as tests that could be simulated were scarce. Some problems found were that the parameters used were not clearly stated, nor the results. I found one experimental test in "The Journal of Fluids and Structures" (2005) that could be simulated by the marine riser program.

3.1 Proofs

3.1.1 Hanging Chain Proof

In this test I was to record the natural frequency of a chain and a riser. The natural frequency of the chain should be close to the value produced by the known formula:

$$w(k)=1/2*r(k)*sqrt(g/LO) \qquad (1)$$

The r values are the roots of the function $J_o(u)=0$ shown in the hanging chain proof in appendix A and K = {1,2,3,4,5…}.

To simulate a chain in the marine riser program, the bending stiffness should be set to zero to simulate the chain links. Also, for a hanging object that is not to touch the ground, the segment stiffness between P1 and P0 should be set to zero. This breaks the connection to the ground to allow free-swinging movement.

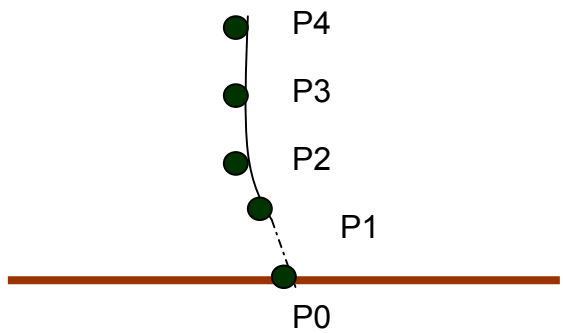


Figure 2 This is a schematic of the lumped masses representing a hanging chain. The top point is pinned and P0 is offset to start a swinging motion from which a natural frequency will arise.

A long riser should also behave similar to a hanging chain. In this test a length of 112m is used. To determine the natural frequency of the riser, the same procedure is to be followed, but with bending stiffness in place.

To calculate the natural frequency, the medium density should be zero to simulate a vacuum. The program ran with this parameter in place but produced very uncontrolled results. Changing the medium to seawater (rhof=1025) produced the expected oscillating behaviour for both the chain and the riser.

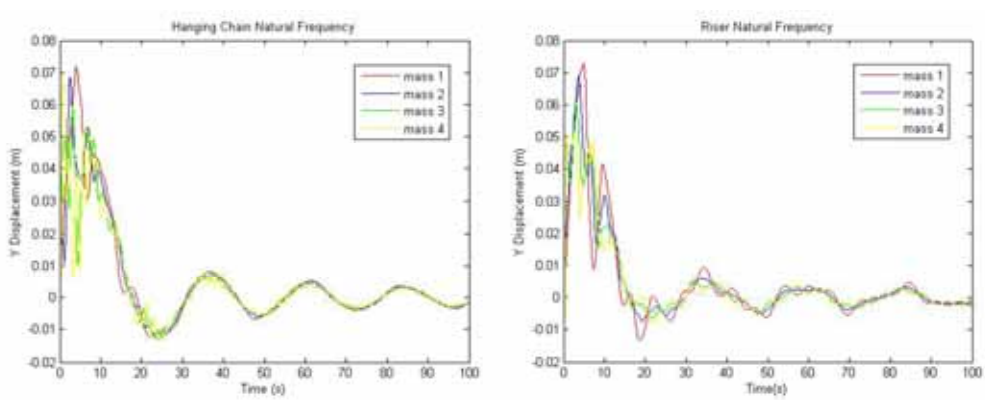


Figure 3 Hanging chain (left) and a riser's (right) natural frequencies.

Table 1 Natural Frequencies of a hanging chain and riser

Riser or Chain	Period (s)	Frequency (hz)	Angular frequency (rad/s)
Chain	24.7	0.0405	0.2545
Riser	20.9	0.0478	0.3003

The formulated values for a hanging chain's natural frequency are $w(1)$=0.35586 rad/s, $w(2)$= 0.816847 rad/s. The values produced from the marine riser program do not compare. I attribute this to the fact that the medium could not be set as air. With medium density, lift and drag in place, the natural frequency of the hanging chain and riser were affected.

3.1.2 Catenary Mooring Proof

My second test was to compare the marine riser program against the known "Elastic Catenary Mooring" proof. This proof states that with given values for the forces H and V, or the values of a and b, a catenary profile can be determined. Dr. Raman-Nair has already written code for these two cases.

These two codes are used in the preliminary steps of setting up the riser parameters.
By estimating H and V and running one of the codes, I was able to produce a profile that I was satisfied with, and then was able to estimate the values of a and b from the graph produced. These values were then used to determine Pn and P0 for the end point positions for the riser simulation. The riser simulated was a steel riser of a 1000m length.

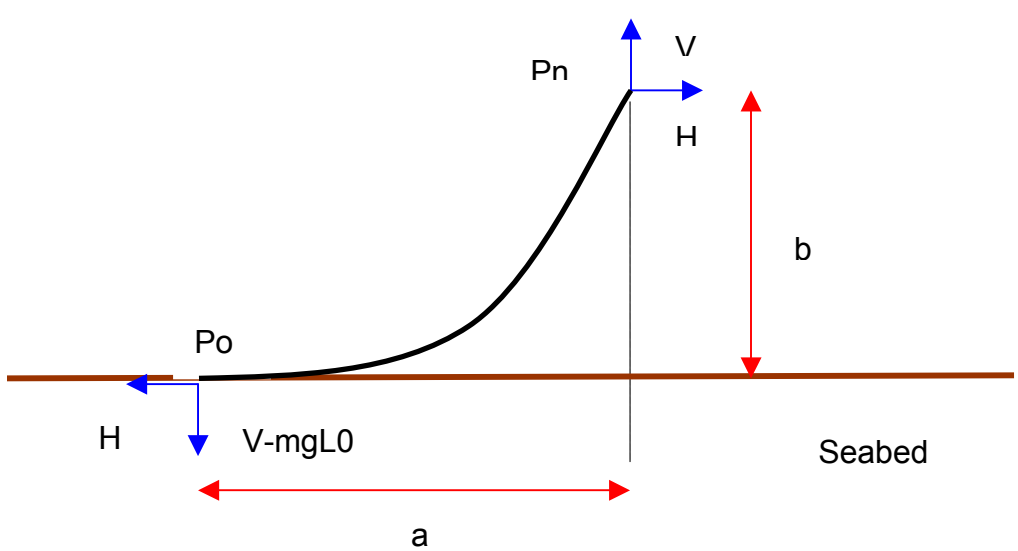


Figure 4 The catenary profile.

After running the marine riser code, I watched the animation of the riser to see if it settled on the catenary profile. Figure 5 shows the final result after the riser had settled. As you can see, it does converge to the catenary profile.

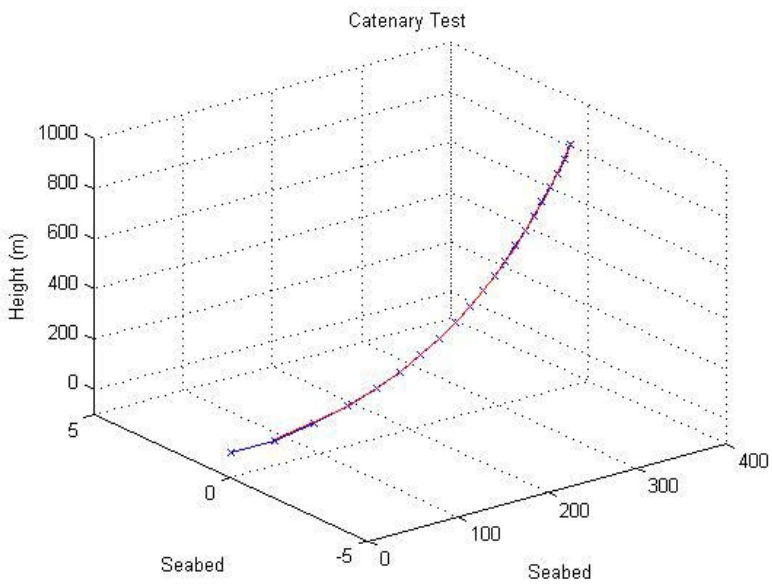


Figure 5 The red line represents the catenary profile while the blue line represents the riser after it has settled.

3.1.3 Harmonic Loading on a Horizontal Beam

My third test was to compare the marine riser program against the known proof "Steady State Response of a Simply Supported Beam Under Uniformly Distributed Harmonic Load", as seen in Appendix A. In this test the beam was oriented horizontally and pinned at both ends. It was placed at a height of 100m so it wouldn't have contact with the 'seabed'. The beam was forced with a sinusoidal load in the z direction.

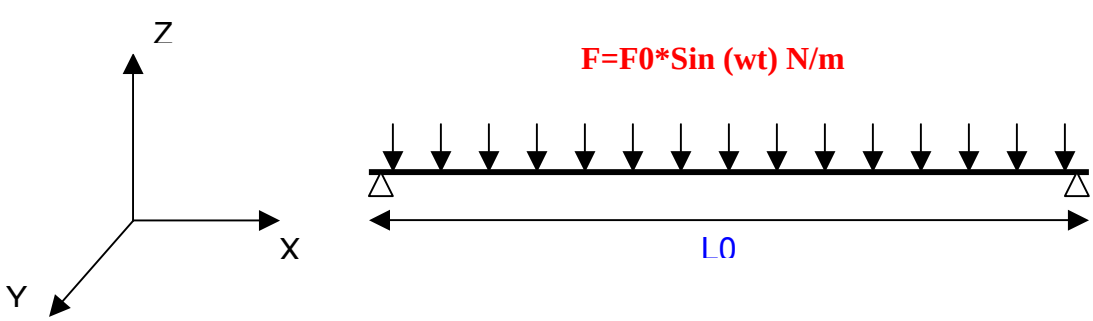


Figure 6 Harmonic loading on a simply supported beam.

The riser simulated was a solid steel riser of 10 m length, solid means there was no inner diameter, so it behaved like a beam. Also, the medium density (rhof) was changed to 0 to simulate the test in air. After the test was performed, a plot was created of time vs. motion in the z direction and the deflection amplitude was measuered for a certain point on the beam and could then be compared to the known formula:

$$v(x) = 2*F0*L0\wedge 4 / (\Pi*E*I)*\{ 2*Sin(\Pi*x/L0) / (\Pi\wedge 4 - omega_bar\wedge 2) + 2*Sin(3*\Pi*x/L0) / (3*((3*\Pi)\wedge 4 - omega_bar\wedge 2)) + 2*Sin(5*\Pi*x/L0) / (5*((5*\Pi)\wedge 4 - omega_bar\wedge 2)) + \dots \} \quad (2)$$

$$omega_bar = omega_beam* sqrt(rho_material *A*L0\wedge 4 / (E*I)) \quad (3)$$

Some problems occurred when setting up the parameters for this test. The damping ratio could not be set to zero, the lowest value it would run at was 0.1. Several tests were performed with zeta=0.3 and zeta=0.1. It didn't seem to affect the final amplitude, just the time it takes for the system to settle out. For zeta=0.3, it settles out in 20s as compared to over 60s for zeta=0.1.

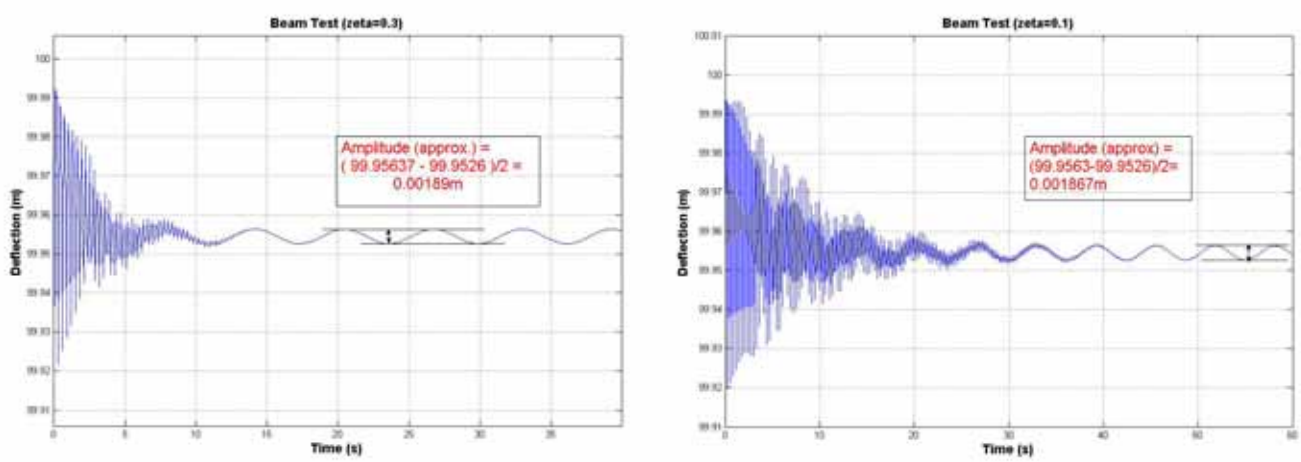


Figure 7 Zeta's affects on deflection amplitude, zeta=0.1 (left) and zeta=0.3 (right).

The formulated amplitude was calculated using a MatLAB m-file I created using formula's (2) and (3). As shown in table 2, the test amplitude doesn't compare to the formulated value, it is about 1/3 the magnitude. We could not set gravity to zero, but I do not believe this has any effect on the resultant amplitude.

Table 2 Test Amplitudes for different zetas

Zeta	Test Amplitude	Formulated Amplitude:
0.1	0.00187	0.006m
0.3	0.00189	

3.2 Experimental Results

Another way to test the marine riser code is to find past experiments that can be simulated and compare the computer generated results to the real results. Reports that can be simulated are hard to find, the parameters in most reports are not stated and the results are also difficult to compare against.

I simulated the parameters from the report "Experimental Investigation of Vortex-Induced Vibration of Long Marine Risers" A.D.Trim et al - Journal of Fluids and Structures (2005). In this study they set up a riser horizontally in a flow tank and set the current velocity to be 0.5 m/s. They then recorded the cross flow oscillations and plotted the displacements.

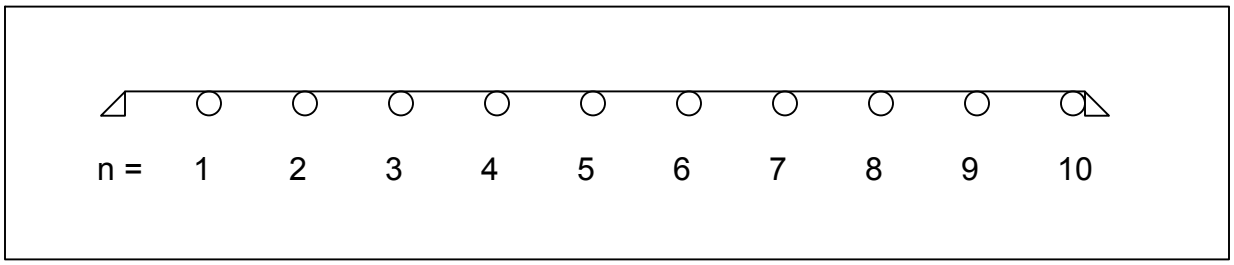


Figure 8 This is a schematic of the riser used in the marine riser program; it is pinned on both ends and is modelled by connected mass lumps.

There was no indication of the drag and lift coefficients for the experimental test so I first used a mean lift of 0.2. The resultant cross flow displacement ranged from approximately ⅓ to ¼ of the diameter. In the experimental test they ranged from approximately ½ to 1 diameter. I increased the lift coefficient to 1.0, to see if my estimate was affecting the results. This new lift coefficient didn't cause any noticeable changes in the displacement.

Next, I tried changing the material density. The riser was made of reinforced fibreglass and no density was stated in the report. I first used a mean value from the range 1300-1800 kg/m^3 [5]. Increasing this value to the upper limit caused the amplitudes to increase and mimic the report more closely.

Table 3 Changing Variables to Simulate Real Test Conditions

Density (kg/m^3)	**Lift**	**Mass 5 (m)**	**Mass 9 (m)**
1500	0.2	0.0075	0.005
	1.0	0.0075	0.005
1800	0.2	0.005	0.015
	1.0	**0.024**	**0.013**

Here are my results for the maximum cross flow displacement between 25 and 30s using a mean lift coefficient of 1.0 and a material density of 1800 kg/m^3:

Table 4 Maximum Deflections of Each Lumped Mass

Mass #	1	2	3	4	5	6	7	8	9
Deflection (m)	0.020	0.022	0.019	0.023	**0.024**	0.018	0.020	0.018	**0.013**

The given diameter was 0.027 m, and as you can see, with the corrected lift and material density, the results vary from approximately ½ to 1 diameter, just as in the experimental results.

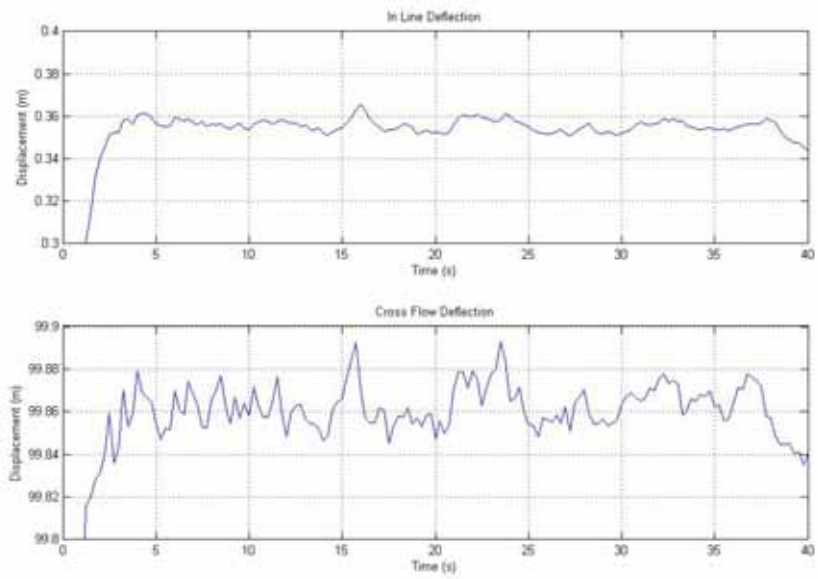


Figure 9 These graphs are shown on the same y-axis scale to show the difference in amplitude.

According to another report by J. Kim Vandiver, "Response Analysis of the of the Flow-Induced Vibration of Flexible Cylinders…" (1981), it was recorded that the in line deflection (IL) amplitude is half that of the cross flow deflection (CF) amplitude but occurs at twice the frequency. In my simulated test, plotting IL and CF in figure 9, you can clearly see that the deflection amplitudes correctly model the findings of that report. Looking at figure 10, the frequencies appear to be of the same magnitude. This is a surprising outcome because the formulas used in the marine riser program for lift and drag differ by this ratio, properly modeling this frequency relationship.

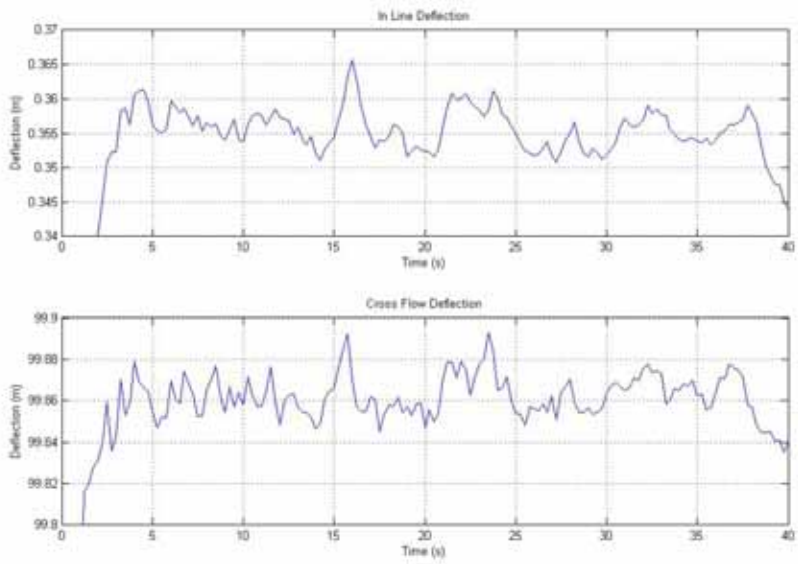


Figure 10 Frequencies seem to be equal when comparing the CF and ILdeflections

4.0 LIFT EXPERIMENT

This test was performed to measure the effect of the lift coefficient on the amplitude of riser vibration. In this test the Bonne Bay experiment was mimicked. This experiment is part of Dr. Bose's physical testing and will occur in Bonne Bay this summer. A steel riser, of 112m length, will be towed behind a boat, simulating a 1 m/s current. The riser also has a 900lb mass on the end to prevent it from floating up. In my testing I reduced the current to 0.5 m/s to cut down on the code run time. The testing still took approx 10 hours per run.

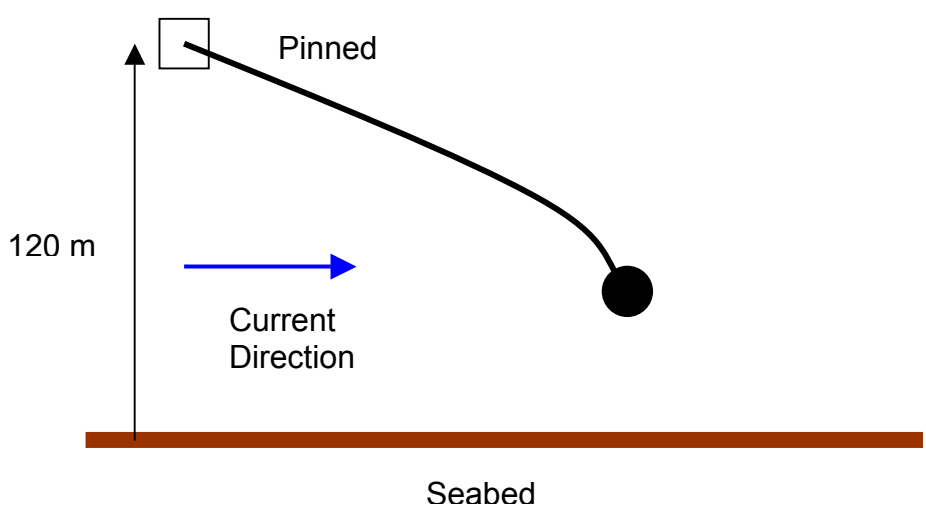


Figure 11 Riser with 900lb mass in 0.5 m/s current.

I varied the lift coefficients from 0.2 to 1.0 at intervals of 0.2 and ran the tests for a time span of [0;0.25;1000]. The results are seen in table 5 for plotting the y vs. time for mass 1, at the tip of the riser, between 700 and 900s. Graphing these points show that amplitude increases with increasing lift.

Table 5 Increasing Lift on Amplitude

Lift	Min (x, y)	Max (x, y)	Amplitude
0.2	(782.6, -0.1240)	(845.1, 0.1290)	0.1265
0.4	(796.8, -0.3863)	(898.8, 0.2658)	0.3261
0.6	(873.5, -0.4788)	(894.8, 0.4424)	0.4606
0.8	(786.8, -0.4833)	(769.0, 0.5693)	0.5263
1.0	(775.5, -0.7708)	(852.3, 0.4583)	0.6146

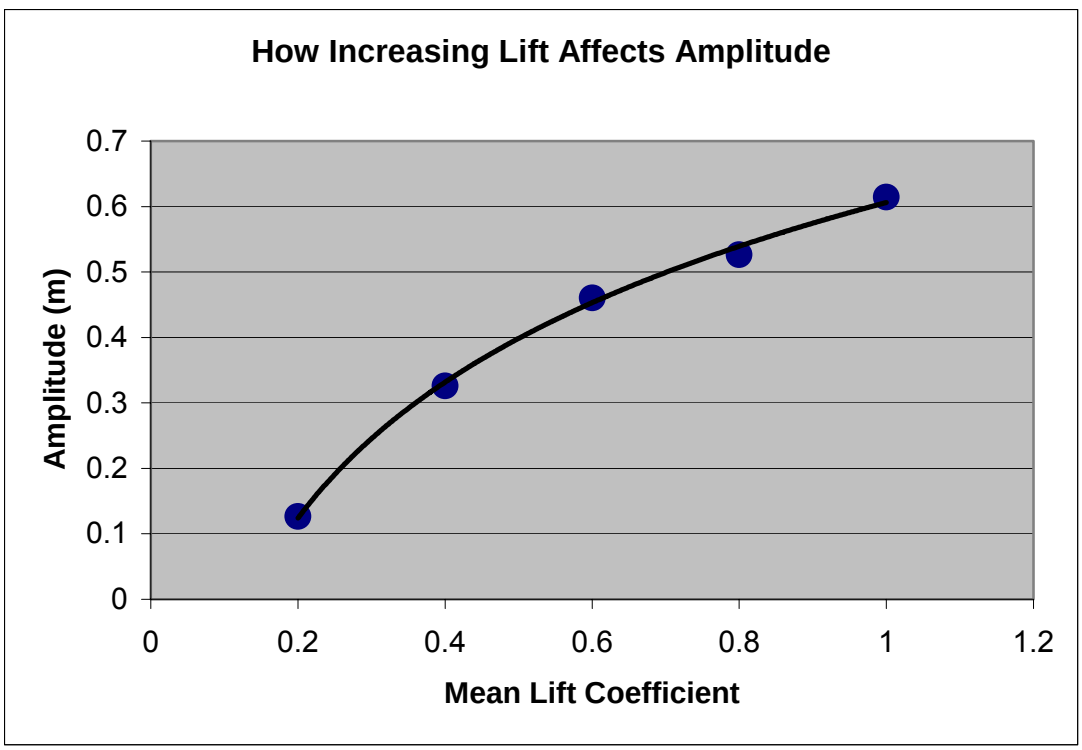


Figure 12 Amplitude magnitude increases with increasing lift.

5.0 LONG RISER TESTING

This test involved simulation a catenary riser with an origional length of 3000m, representing the realistic length for a production riser. It is placed in a stepped current where the current varies from 0 m/s to 1 m/s as shown in Figure 13. This riser was broken in to 100 lumped masses, each mass being 300 meters apart.

In this simulation I varied the lift coefficient (CL0) from 0.2 to 1.0 in steps of 0.2 and then measured the maximum tension and maximum bending moment in the riser.

The maximum tension always occurred at the topmost point, n=100, and the maximum bending moment always occurred at the lowest point, at n=1. Since I only used 100 lumped masses in my testing, these locations are not precise. The values recorded are to be used to show the general effect of increasing lift on the maximum tension and bending moments. All values are recorded between 300 and 1000s to allow time for the system to settle out.

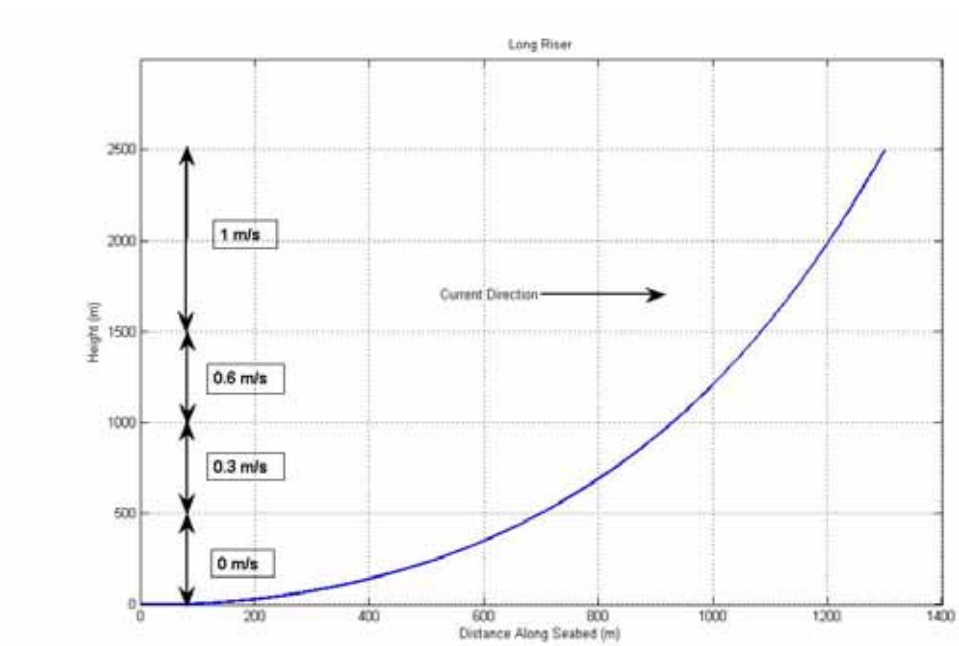


Figure 13 Testing conditions for long catenary riser simulation.

5.1 Lift Coefficients

5.1.1 Lift Coefficient of 0.2

Mean Min. Tension 1.97677e6 N
Mean Max. Tension 1.08353e7 N

- Maximum Point 1.08375e7 N
- Minimum Point 1.08334e7 N
- Max. Amplitude 2050 N
- Frequency 0.35 Hz

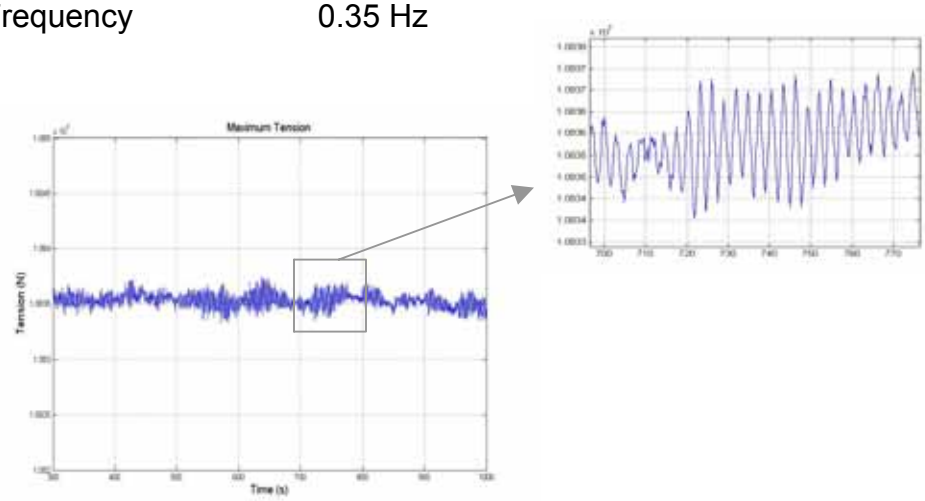


Figure 14 Maximum tension for lift coefficient of 0.2.

Mean Min. Bending Moment 2.7389e4 m*kg
Mean Max. Bending Moment 7.92725e5 m*kg

- Maximum Point 7.9315e5 m*kg
- Minimum Point 7.92025e5 m*kg
- Max Amplitude 563 m*kg
- Frequency 0.15 Hz – 0.3Hz

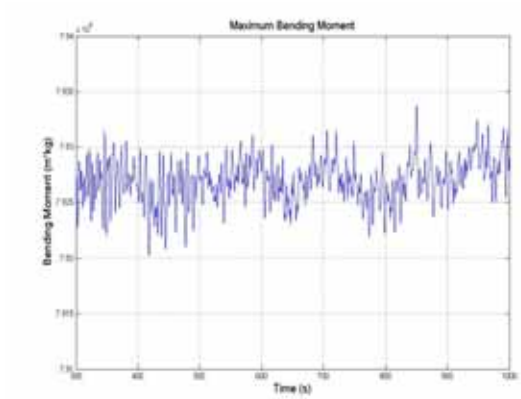


Figure 15 Maximum bending moment for the lift coefficient of 0.2.

5.1.2 Lift Coefficient of 0.4

Mean Min. Tension	1.97585e6 N
Mean Max. Tension	1.08344e7 N
• Maximum Point*	1.08371e7 N
• Minimum Point*	1.08317e7 N
• Max Amplitude	2700 N
• Tension Frequency	0.35 Hz

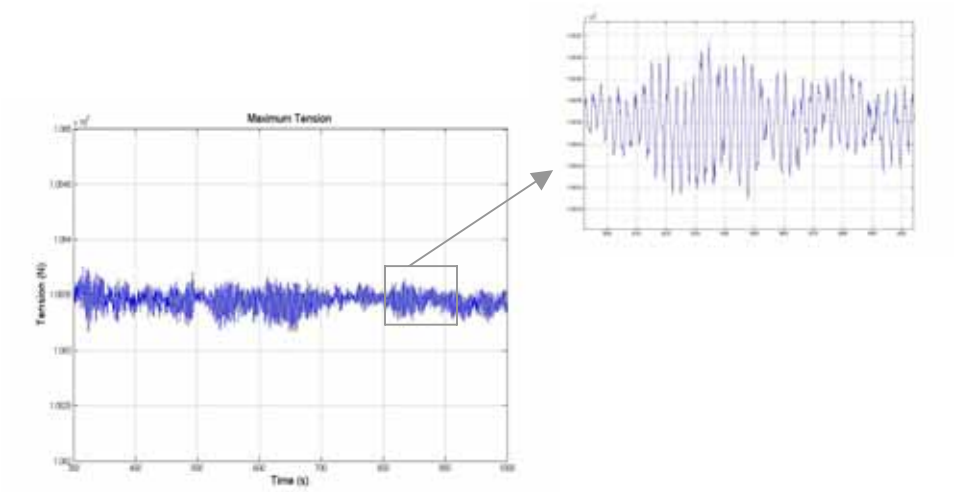


Figure 16 Maximum Tension for mean lift coefficient of 0.4.

Mean Min. Bending Moment	2.75e4 kg*m
Mean Max. Bending Moment	7.92975e5 m*kg
• Maximum Point*	7.93575e5 m*kg
• Minimum Point*	7.92285e5 m*kg
• Max Amplitude	645 m*kg
• Frequency	0.15 Hz

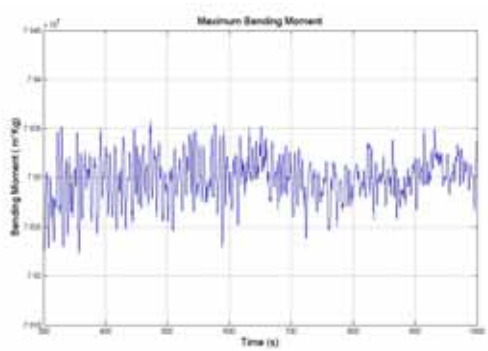


Figure 17 Maximum bending moment for mean lift coefficient of 0.4.

5.1.3 Lift Coefficient of 0.6

Mean Min. Tension	1.975e6 N
Mean Max. Tension	1.08336e7 N
• Maximum Point	1.0837e7 N
• Minimum Point	1.083075e7 N
• Max Amplitude	3125 N
• Frequency	0.35 Hz

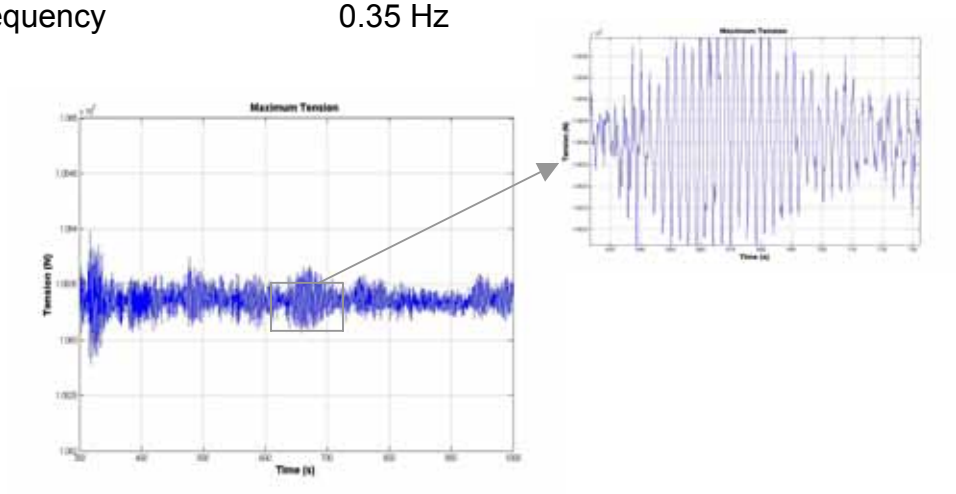


Figure 18 Maximum tension for mean lift coefficient of 0.6.

Mean Min. Bending Moment	2.76e4 kg*m
Mean Max. Bending Moment	7.93315e5 kg*m
• Maximum Point	7.94015e5 kg*m
• Minimum Point	7.9264e5 kg*m
• Max Amplitude	688 kg*m
• Frequency	0.15 Hz – 0.225Hz

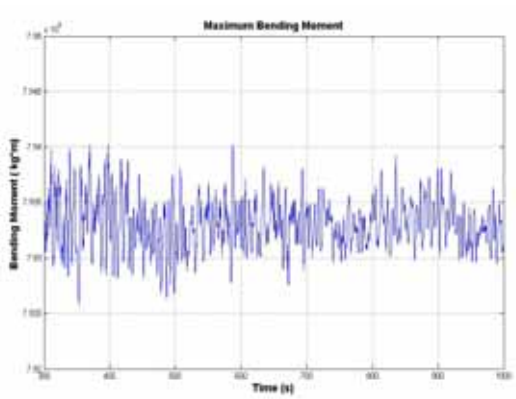


Figure 19 Maximum bending moment for mean lift coefficient of 0.6.

5.1.4 Lift Coefficient of 0.8

Mean Min. Tension	1.97438e6 N
Mean Max. Tension	1.08329e7 N
• Maximum Point	1.08375e7 N
• Minimum Point	1.08288e7 N
• Max. Amplitude	4350 N
• Frequency	0.35 Hz

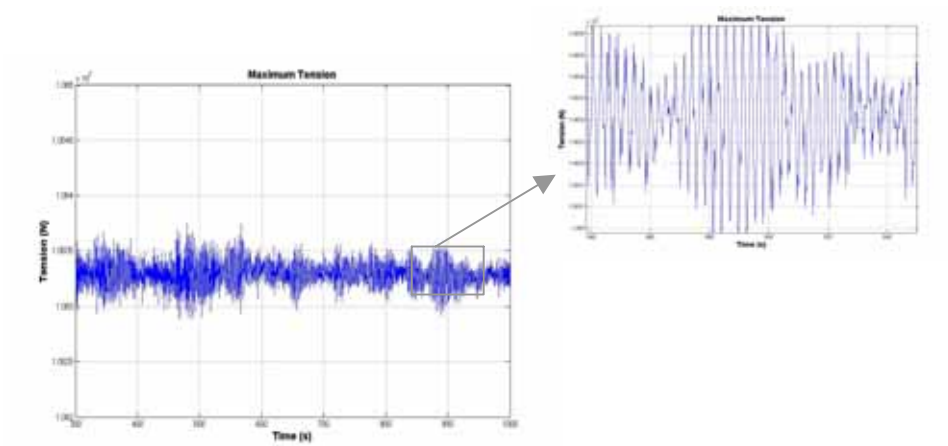


Figure 20 Maximum tension for mean lift coefficient 0f 0.8.

Mean Min. Bending Moment	2.7848e4 kg*m
Mean Max. Bending Moment	7.93535e5 kg*m
• Maximum Point	7.94250e5 kg*m
• Minimum Point	7.92825e5 kg*m
• Max. Amplitude	713 kg*m
• Frequency	0.2 Hz

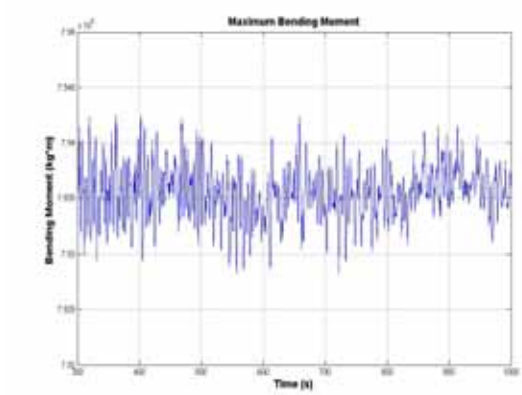


Figure 21 Bending moment time history for lift coefficient of 0.8.

5.1.5 Lift Coefficient of 1.0

Mean Min. Tension	1.97355e6 N
Mean Max. Tension	1.08324e7 N
• Maximum Point	1.08369e7 N
• Minimum Point	1.08275e7 N
• Max. Amplitude	4700 N
• Frequency	0.4 Hz – 0.5Hz

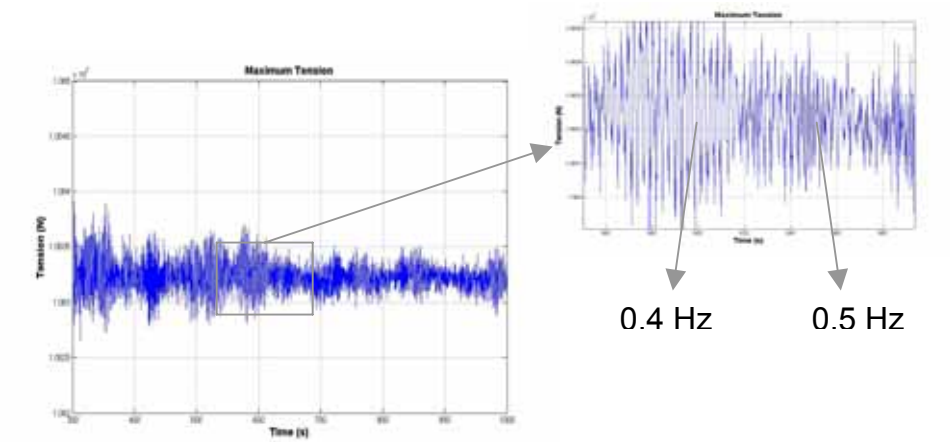


Figure 22 Maximum tension for mean lift coefficient of 1.0.

Mean Min. Bending Moment	2.785e4 kg*m
Mean Max. Bending Moment	7.93755e5 kg*m
• Maximum Point	7.9461e5 kg*m
• Minimum Point	7.9313e5 kg*m
• Max. Amplitude	740 kg*m
• Frequency	0.2 Hz – 0.3Hz

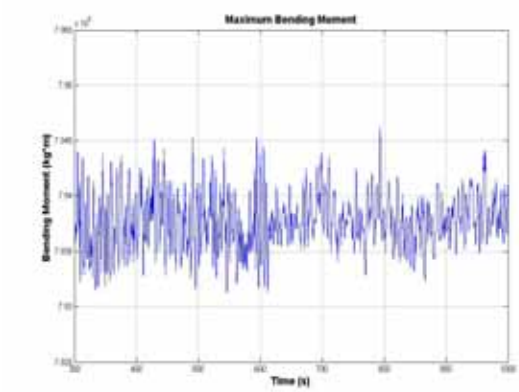


Figure 23 Maximum bending moment for mean lift coefficient of 1.0.

5.2 Parameter Relationships

From these results, it can be said that increasing lift does have a significant effect on the resultant maximum values and amplitudes of both the tension and bending moment. The changes in magnitudes were not so significant, but definite relationships can be seen. IN figure 24, it is seen that maximum tension decreases with increasing lift. It has the opposite effect on bending moment, as it increases with increasing lift, as shown in figure 26. Both the amplitudes, for tension and bending moment, increased with increasing lift. The frequencies remained about the same but became increasingly harder to calculate as the lift increased. Sometimes more than one frequency existed for a particular graph, as shown in figure 22.

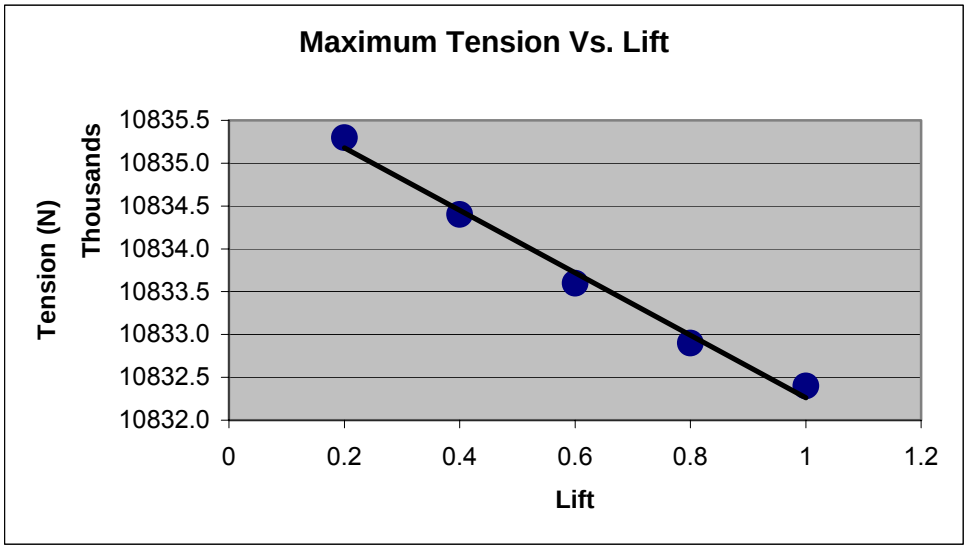


Figure 24 Max. Tension decreases with increasing lift coefficient.

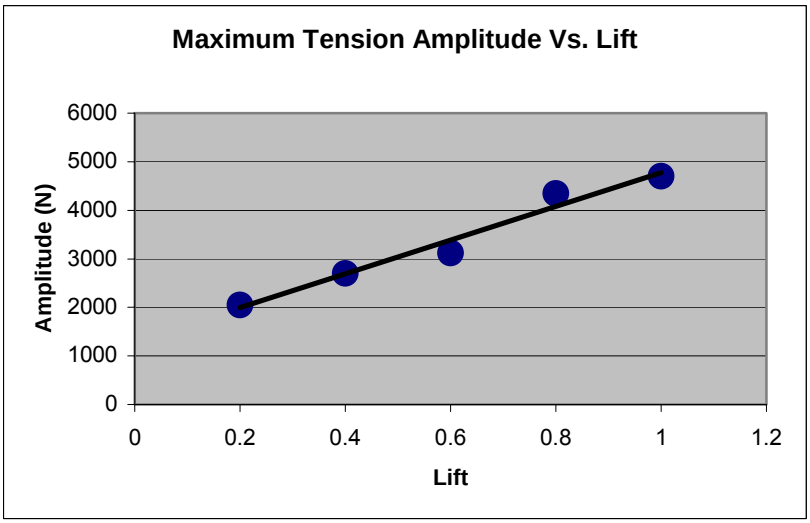


Figure 25 Tension amplitude increases with increasing lift coefficient.

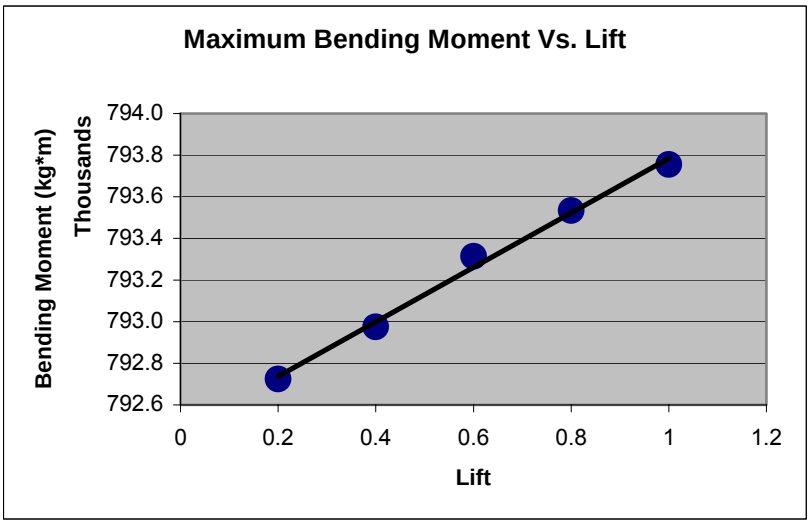


Figure 26 Maximum bending moment increases with increasing lift.

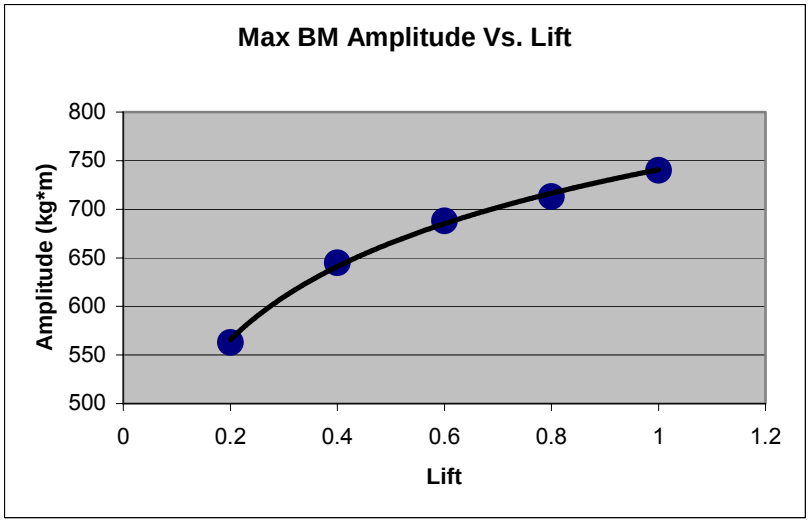


Figure 27 Maximum bending moment amplitude increases with increasing lift.

6.0 CONCLUSION AND RECOMMENDATIONS

These are the tests that produced correct results, properly simulating riser dynamics:

- Catenary mooring proof
- Trim et al (2005 Journal of Fluids and Structures)

These tests produced bad results:

- Hanging chain
- Harmonic loading of a solid beam

In my testing, there were many obstacles that had to be overcome in simulating these tests. Sometimes the correct setup could not be modelled by the program, due to the assumptions made when developing the proofs, and some parameters had to be changed. This caused a disagreement between the computer generated results and the formulated ones.

The ability for the program to simulate a real experiment and produce comparable results is a very encouraging finding but it is very difficult to find tests that clearly state their setup parameters and/or results clearly. I recommend that we set up our own tests to compare the marine riser program against. This way the testing parameters are known and specific properties can be targeted.

7.0 REFERENCES

1. MUN Faculty of Engineering Grants and Awards, http://www.engr.mun.ca/research/grants.php

2. Dr. I Taylor: 16598, Aeroelasticity (slide show)

3. Ali H. Nayfeh et al, Modeling the transient and steady-state flow over a stationary cylinder (DET2005-85376), ASME, 2005

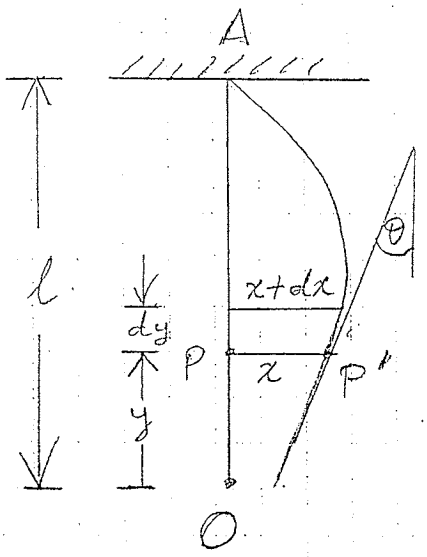
4. Trim et al, Experimental investigation of vortex induced vibration of long marine risers (335-361), Journal of Fluids and Structures, Elsevier Ltd. 2005

5. FRP design considerations, http://www.fiberbond.com/docs/FRPdesign.pdf

APPENDIX A

The Hanging Chain.

Ref : Bessel Functions for Engineers
by N. W. Mc Lachlan, 2nd ed.
Oxford University Press, 1955.



Chain AO is suspended from fixed point A, with end O free.

Mass per unit length $= \rho$.

Length of chain $= l$.

Consider point P at distance y from O as shown and let the horizontal deflection at P be $x(y,t)$ where t is time.

For small deflection x, the tension $T(y)$ at P is approximately equal to the weight of OP

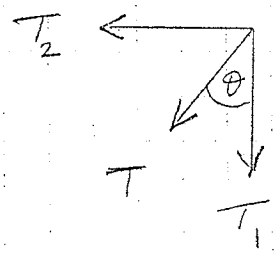
i.e. $T(y) = \rho g y$ (= mg) — (1).

After small deflection, P is at P'. Let the tangent at P' make angle θ with the y axis.

Then $\tan\theta = \dfrac{\partial x}{\partial y}$

For small θ, $\sin\theta \approx \tan\theta = \dfrac{\partial x}{\partial y}$.

Components of $T(y)$:



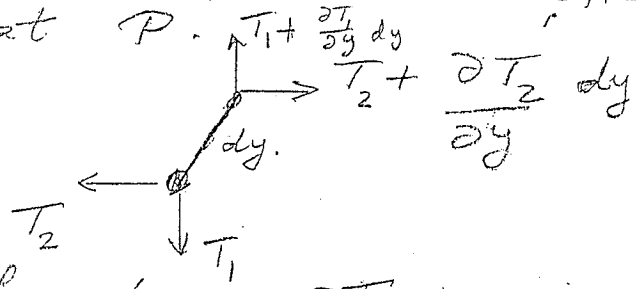
$T_2 = T \sin\theta$

$T_1 = T \cos\theta$.

$$T_2 = T \sin\theta = T \frac{\partial x}{\partial y} = \rho g y \frac{\partial x}{\partial y} \quad \text{--- (2)}$$

using (1).

We consider horizontal motion of an element dy at P.



$$\frac{\partial T_1}{\partial y} \approx 0$$

Eqn. of motion: (x direction)

$$\frac{\partial T_2}{\partial y} dy = \rho\, dy \frac{\partial^2 x}{\partial t^2} \rightarrow$$

$$\frac{\partial T_2}{\partial y} = \rho \frac{\partial^2 x}{\partial t^2} \quad \text{--- (3)}$$

From (2):

$$\frac{\partial T_2}{\partial y} = \rho g \frac{\partial}{\partial y}\left(y \frac{\partial x}{\partial y}\right).$$

Subst. in (3):

$$\rho g \frac{\partial}{\partial y}\left(y \frac{\partial x}{\partial y}\right) = \rho \frac{\partial^2 x}{\partial t^2}$$

i.e.

$$\frac{\partial^2 x}{\partial t^2} = g \frac{\partial}{\partial y}\left(y \frac{\partial x}{\partial y}\right) \quad \text{--- (4)}$$

or

$$\frac{\partial^2 x}{\partial t^2} = g\left(y \frac{\partial^2 x}{\partial y^2} + \frac{\partial x}{\partial y}\right). \quad \text{--- (5)}$$

Assume that $z(y,t) = f(y)\,h(t)$

where f and h are twice differentiable functions of y and t respectively. Subst in (5):

$$f\frac{d^2h}{dt^2} = g\left(yh\frac{d^2f}{dy^2} + h\frac{df}{dy}\right)$$

Divide by fh:

$$\frac{1}{h}\frac{d^2h}{dt^2} = g\left(\frac{y}{f}\frac{d^2f}{dy^2} + \frac{1}{f}\frac{df}{dy}\right)$$

The L.H.S. is independent of z and the R.H.S. is independent of z

Since equality holds for $t > 0$, $0 \le z \le l$, each side must be equal to a constant, say $-\omega^2$ i.e.

$$\frac{1}{h}\frac{d^2h}{dt^2} = -\omega^2 \quad \text{--- (6)}$$

$$g\left(\frac{y}{f}\frac{d^2f}{dy^2} + \frac{1}{f}\frac{df}{dy}\right) = -\omega^2 \quad \text{--- (7)}$$

From (6): $\dfrac{d^2h}{dt^2} + \omega^2 h = 0$

$$\Rightarrow \quad h(t) = A\sin(\omega t) + B\cos(\omega t) \quad \text{--- (8)}$$

where A, B are arbitrary constants.

From (7):

$$y\frac{d^2f}{dy^2} + \frac{df}{dy} + \frac{\omega^2 f}{g} = 0 \quad \text{- - - -} \quad (9)$$

Let $u^2 = \alpha y$ where $\alpha = \frac{4\omega^2}{g}$

Then $2u\,du = \alpha\,dy$

$$\Rightarrow \frac{du}{dy} = \frac{\alpha}{2u}$$

$$\frac{df}{dy} = \frac{df}{du}\cdot\frac{du}{dy} = \frac{\alpha}{2u}\frac{df}{du} \quad \text{- - -} \quad (i)$$

$$\frac{d^2f}{dy^2} = \frac{d}{dy}\left(\frac{df}{dy}\right) = \frac{d}{du}\left(\frac{df}{dy}\right)\frac{du}{dy}$$

$$= \frac{\alpha}{2u}\cdot\frac{d}{du}\left(\frac{\alpha}{2u}\frac{df}{du}\right) = \frac{\alpha^2}{4u}\cdot\frac{d}{du}\left(\frac{1}{u}\frac{df}{du}\right)$$

$$= \frac{\alpha^2}{4u}\left(\frac{1}{u}\frac{d^2f}{du^2} - \frac{1}{u^2}\frac{df}{du}\right)$$

$$y\frac{d^2f}{dy^2} = \frac{u^2}{\alpha}\cdot\frac{\alpha^2}{4u}\left(\frac{1}{u}\frac{d^2f}{du^2} - \frac{1}{u^2}\frac{df}{du}\right)$$

$$= \frac{\alpha}{4}\left(\frac{d^2f}{du^2} - \frac{1}{u}\frac{df}{du}\right) \quad \text{- - - -} \quad (ii)$$

Subst. (i), (ii) into (9):

$$\frac{\alpha}{4}\left(\frac{d^2f}{du^2} - \frac{1}{u}\frac{df}{du}\right) + \frac{\alpha}{2u}\frac{df}{du} + \frac{\alpha f}{4} = 0.$$

Multiply by $\frac{4}{\alpha}$:

$$\frac{d^2f}{du^2} + \frac{1}{u}\frac{df}{du} + f = 0 \quad \text{- - -} \quad (10)$$

Eqn (10) is Bessel's differential equation of order zero. The general solution is

$$f(u) = A_0 J_0(u) + B_0 Y_0(u) \quad \text{---} \quad (11)$$

where $J_0(u)$ is the Bessel function of the first kind of order zero

$Y_0(u)$ is the Bessel function of the second kind of order zero. (weber function)

At $y = 0$, $u = 0$ and $Y_0 = -\infty \Rightarrow B_0 = 0$.

$u^2 = \alpha y \rightarrow u = \sqrt{\alpha y}$

$$\therefore \quad f(u) = A_0 J_0(u) = A_0 J_0(\sqrt{\alpha y}) \quad \text{---} \quad (12)$$

where A_0 is an arbitrary constant

The general solution is

$$x(y, t) = J_0(\sqrt{\alpha y})\left(A \cos(\omega t) + B \sin(\omega t)\right) \quad \text{---} \quad (13)$$

$$\text{where} \quad \alpha = \frac{4\omega^2}{g}.$$

Standing wave solutions are obtained from the boundary condition $x(l, t) = 0 \quad \forall t$.

From (13): $J_0(\sqrt{\alpha l}) = 0 \quad \text{---} \quad (14)$

Let the roots of $J_0(u) = 0$ be $r_1, r_2, \ldots$

Corresponding to r_k, we have $\sqrt{\alpha_k l} = r_k$.

$$\Rightarrow \quad \alpha_k = \frac{r_k^2}{l} \quad \text{---} \quad (15)$$

If the corresponding value of ω is ω_k,

$$\frac{4\omega_k^2}{g} = \frac{r_k^2}{l} \Rightarrow \omega_k = \frac{1}{2} r_k \sqrt{\frac{g}{l}} \quad \text{---} \quad (16)$$

root

$\omega = 2\pi f$

The corresponding solution for x is (kth mode) (6)

$$x_k(y,t) = J_0\left(r_k\sqrt{\frac{y}{l}}\right)\left\{A_k \cos(\omega_k t) + B_k \sin(\omega_k t)\right\}$$

$(k = 1, 2, \ldots$

Roots of $J_0(u) = 0$ (First 6 roots) — — — (17)

$r_1 = 2.404825558$ $\quad$ $r_2 = 5.520078110$

$r_3 = 8.653727913$ $\quad$ $r_4 = 11.79153444$

$r_5 = 14.93091771$ $\quad$ $r_6 = 18.07106397.$

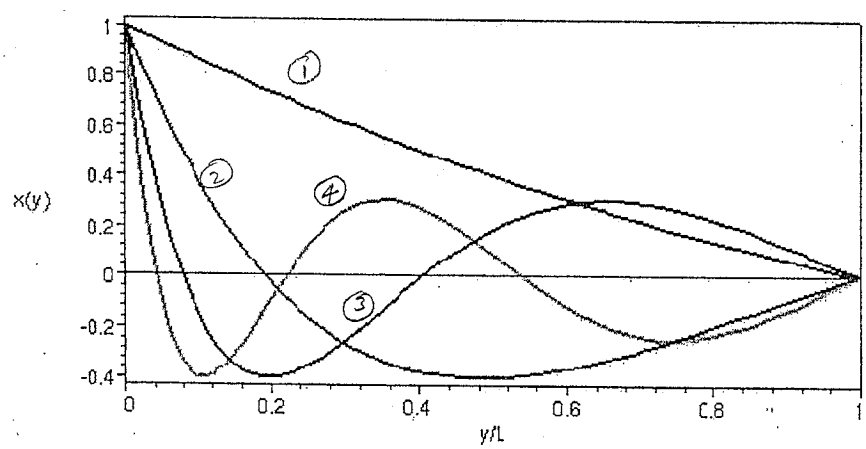


Figure 3.2: The first 4 standing wave modes of a hanging cable.

Amplitudes

$$x_k(y,t) = J_0\left(r_k\sqrt{\frac{y}{l}}\right)\left\{A_k \cos(\omega_k t) + B_k \sin(\omega_k t)\right\}$$

$$(k = 1, 2, \ldots) \qquad \text{--- (17)}$$

$$\dot{x}_k(y,t) = J_0\left(r_k\sqrt{\frac{y}{l}}\right)\left\{-\omega_k A_k \sin(\omega_k t) + \omega_k B_k \cos(\omega_k t)\right\} \qquad \text{--- (18)}$$

At $t = 0$: $x_k(y,0) = \alpha_k(y)$? $(0 \le y \le l)$.

$$\dot{x}_k(y,0) = 0 \quad \text{for all } y.$$

From (17):

$$\alpha_k(y) = J_0\left(r_k\sqrt{\frac{y}{l}}\right) A_k$$

$$\Rightarrow \quad A_k = \frac{\alpha_k(y)}{J_0\left(r_k\sqrt{\frac{y}{l}}\right)} \qquad (k = 1, 2, \ldots)$$

From (18):

$$0 = \omega_k B_k J_0\left(r_k\sqrt{\frac{y}{l}}\right) \Rightarrow B_k = 0 \text{ for all } k$$

Hence

$$x_k(y,t) = A_k J_0\left(r_k\sqrt{\frac{y}{l}}\right)\cos(\omega_k t)$$

$$= \alpha_k(y)\cos(\omega_k t).$$

Ref.: 'Cable Structures' by H. M. Irvine
MIT Press, 1981, pg. 16.

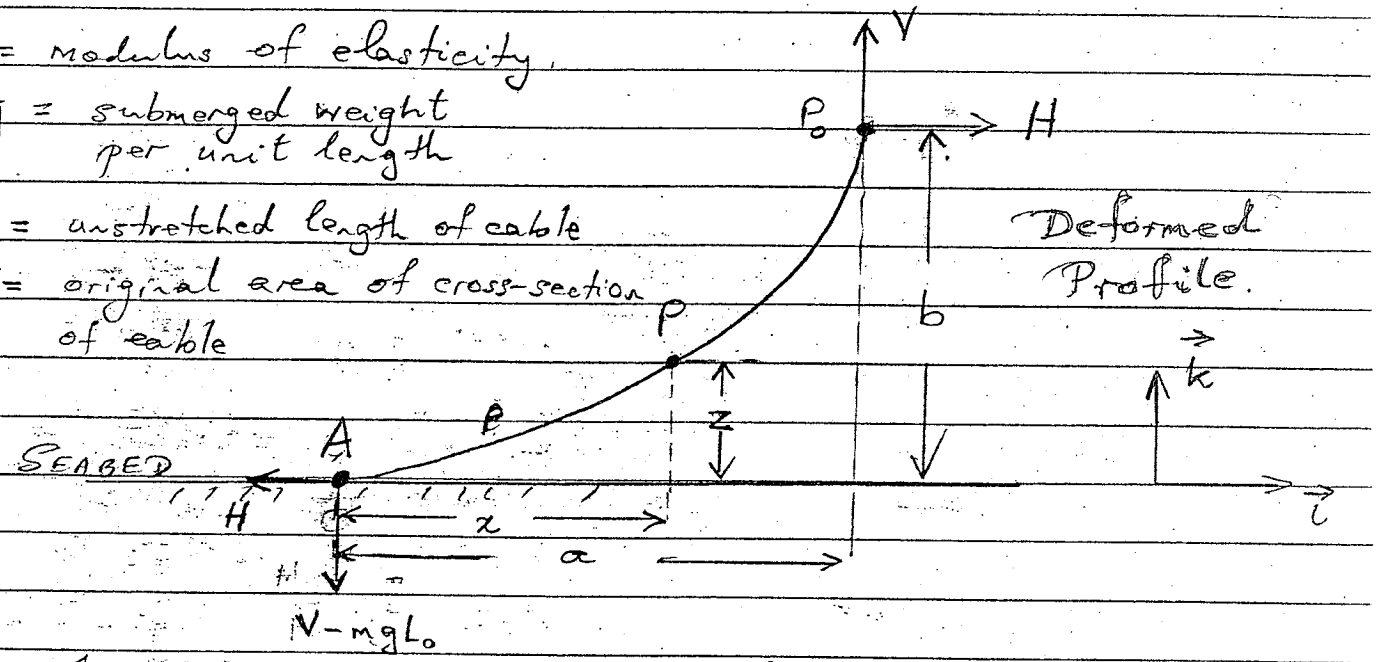
Elastic Catenary Mooring

E = modulus of elasticity

mg = submerged weight per unit length

L_o = unstretched length of cable

A_o = original area of cross-section of cable



The diagram shows the deformed shape of a uniform elastic cable anchored at A and suspended at P_o by supporting forces H and V as shown. Total deformed length = L

The arc length from A to a general point P(x, z) is ρ (deformed shape).

Let s = arc length from A to P when the cable is unstretched.

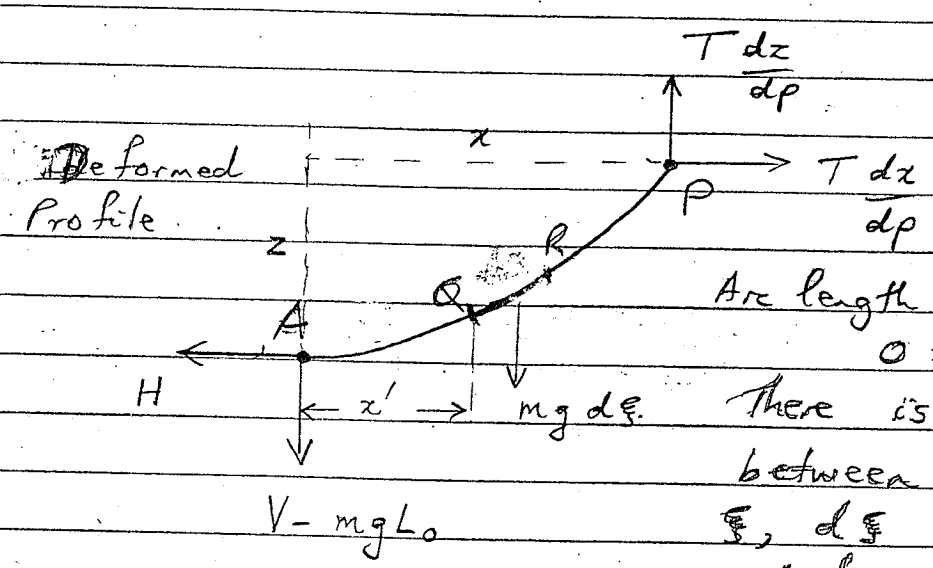
Unit tangent vector ~~at~~ P is (in the sense of increasing ρ)

$$\vec{t} = \frac{dx}{d\rho}\vec{i} + \frac{dz}{d\rho}\vec{k} \quad \ldots \quad (1)$$

$$|\vec{t}| = 1 \Rightarrow \left(\frac{dx}{d\rho}\right)^2 + \left(\frac{dz}{d\rho}\right)^2 = 1 \quad \ldots \quad (2)$$

Let the tension at P be T and consider the equilibrium of AP.

Let QR be a small element of AP such that the <u>undeformed</u> arc lengths AQ, QR are ξ, $d\xi$ respectively.



Arc length $AP = \rho$

$0 \le \xi \le s$

There is a 1-1 map between s and ρ.

ξ, $d\xi$ refer to the <u>undeformed</u> profile.

Horizontal Equilibrium:

$$T\frac{dx}{d\rho} = H \quad \text{---} \quad (3)$$

Vertical Equilibrium:

$$T\frac{dz}{d\rho} = \int_0^s mg\,d\xi + V - mgL_0$$

i.e. $$T\frac{dz}{d\rho} = V - mgL_0 + mgs \quad \text{---} \quad (4).$$

Moments about A:

$$\left(T\frac{dx}{d\rho}\right)z + \int_0^s mg\,x'(\xi)\,d\xi = \left(T\frac{dz}{d\rho}\right)x \quad \text{---} \quad (5)$$

We show that (3) and (4) ensure that (5) is automatically satisfied.

Take moments about P

$$Hz = (V - mgL_0)x + \int_0^s mg\,(x - x'(\xi))\,d\xi. \quad \text{---} \quad (6)$$

From ③, ④ subst. for H and $(V - mgL_0)$ in ②

$$\left(T\frac{dx}{dp}\right) z = \left(T\frac{dz}{dp} - mgs\right) x + mgx\int_0^s d\xi - \int_0^s mg\, x'(\xi)\, d$$

$$= \left(T\frac{dz}{dp}\right) x - \int_0^s mg\, x'(\xi)\, d\xi$$

which is the same as ⑤.

<u>Hooke's Law</u>

Strain at P is $\lim_{\Delta s \to 0}\left(\frac{\Delta P - \Delta s}{\Delta s}\right) = \frac{dp}{ds} - 1$

Hence $T = EA_0\left(\frac{dp}{ds} - 1\right)$ - - - - ⑦.

<u>End Conditions</u>

At $s = 0$; $x = 0$, $z = 0$, $p = 0$

At $s = L_0$: $x = a$, $z = b$, $p = L$ (stretched length)

Square and add ③ and ④ and use ②:

$$T^2 = H^2 + (V - mgL_0 + mgs)^2$$

$$T = \left[H^2 + (V - mgL_0 + mgs)^2\right]^{\frac{1}{2}} \quad - - - ⑧.$$

$$\frac{dx}{ds} = \frac{dx}{dp}\cdot\frac{dp}{ds} = \left(\frac{H}{T}\right)\frac{dp}{ds} \quad \text{from } (3)$$

$$= \frac{H}{T}\left(\frac{T}{EA_0} + 1\right) \quad \text{from } (7).$$

Using (8) this becomes

$$\frac{dx}{ds} = \frac{H}{EA_0} + \frac{H}{\left[H^2 + (V - mgL_0 + mgs)^2\right]^{\frac{1}{2}}} \quad \text{--- (9)}$$

If $u = V - mgL_0 + mgs$, $du = mg\,ds$

$$\int \frac{ds}{\left[H^2 + (V - mgL_0 + mgs)^2\right]^{\frac{1}{2}}} = \frac{1}{mg}\int \frac{du}{(H^2 + u^2)^{\frac{1}{2}}}$$

$$= \frac{1}{mg}\sinh^{-1}\left(\frac{u}{H}\right) + \text{constant}$$

$$= \frac{1}{mg}\sinh^{-1}\left(\frac{V - mgL_0 + mgs}{H}\right) + \text{constant}$$

Hence (9) integrates to

$$x(s) = \frac{Hs}{EA_0} + \frac{H}{mg}\sinh^{-1}\left(\frac{V - mgL_0 + mgs}{H}\right) + C$$

At $s = 0$, $x = 0 \Rightarrow$

$$0 = \frac{H}{mg}\sinh^{-1}\left(\frac{V - mgL_0}{H}\right) + C$$

Hence

$$x(s) = \frac{Hs}{EA_0} + \frac{H}{mg}\left\{\sinh^{-1}\left(\frac{V - mgL_0 + mgs}{H}\right) - \sinh^{-1}\left(\frac{V - mgL_0}{H}\right)\right\}$$

--- (10)

$$\frac{dz}{ds} = \frac{dz}{dp} \cdot \frac{dp}{ds} = \left(\frac{V - mgL_0 + mgs}{T}\right)\frac{dp}{ds} \quad \text{from } (4)$$

$$= \left(\frac{V - mgL_0 + mgs}{T}\right)\left(\frac{T}{EA_0} + 1\right) \quad \text{from } (7)$$

$$= \frac{V - mgL_0 + mgs}{EA_0} + \frac{V - mgL_0 + mgs}{\left[H^2 + (V - mgL_0 + mgs)^2\right]^{\frac{1}{2}}} \quad \text{--- } (11)$$

using (8).

Let $u^2 = H^2 + (V - mgL_0 + mgs)^2$

$$2u\,du = 2mg\,(V - mgL_0 + mgs)\,ds.$$

$$\Rightarrow \left(\frac{1}{mg}\right) u\,du = (V - mgL_0 + mgs)\,ds$$

$$\therefore \int \frac{(V - mgL_0 + mgs)\,ds}{\left[H^2 + (V - mgL_0 + mgs)^2\right]^{\frac{1}{2}}} = \frac{1}{mg}\int \frac{u\,du}{u}$$

$$= \frac{u}{mg} + \text{const} = \frac{1}{mg}\left[H^2 + (V - mgL_0 + mgs)^2\right]^{\frac{1}{2}} + \text{constant}.$$

Integrate (11):

$$z(s) = \frac{1}{EA_0}\left[(V - mgL_0)s + \frac{1}{2}mg\,s^2\right]$$

$$+ \frac{1}{mg}\left[H^2 + (V - mgL_0 + mgs)^2\right]^{\frac{1}{2}} + D.$$

At $s = 0$, $z = 0 \Rightarrow$

$$0 = \frac{1}{mg}\left[H^2 + (V - mgL_0)^2\right]^{\frac{1}{2}} + D.$$

Hence

$$z(s) = \frac{s}{EA_0}\left[V - mgL_0 + \frac{1}{2}mgs\right]$$

$$+ \frac{H}{mg}\left\{\left[1 + \left(\frac{V - mgL_0 + mgs}{H}\right)^2\right]^{\frac{1}{2}} - \left[1 + \left(\frac{V - mgL_0}{H}\right)^2\right]^{\frac{1}{2}}\right\}$$

---- (12)

<u>Solutions for H and V.</u>

At $s = L_0$; $x = a$, $z = b$.
Subst. in (10), (12):

$$a = \frac{HL_0}{EA_0} + \frac{H}{mg}\left\{\sinh^{-1}\left(\frac{V}{H}\right) - \sinh^{-1}\left(\frac{V - mgL_0}{H}\right)\right\}.$$

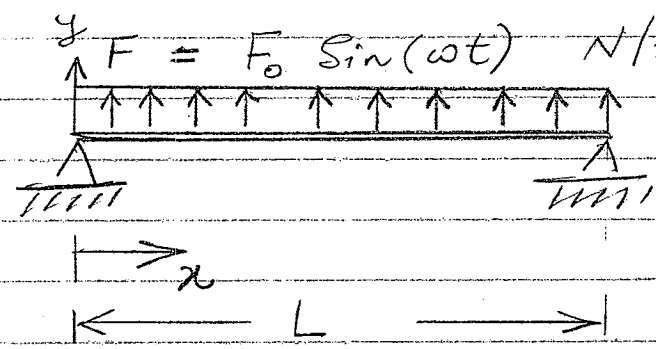
--- (13)

$$b = \frac{L_0}{EA_0}\left(V - \frac{1}{2}mgL_0\right)$$

$$+ \frac{H}{mg}\left\{\left(1 + \frac{V^2}{H^2}\right)^{\frac{1}{2}} - \left[1 + \left(\frac{V - mgL_0}{H}\right)^2\right]^{\frac{1}{2}}\right\}.$$

--- (14).

Steady-State Response of a Simply-Supported Beam Under Uniformly Distributed Harmonic Load.



$F = F_0 \sin(\omega t)$ N/m

mod of elast. — moment of Inertia

E, A, ρ, I

E : N/m^2 $\quad A$: m^2

I : m^4 $\quad \rho$: kg/m^3.

$v(x,t) =$ y-direction deflection

Governing Equation

$$EI \frac{\partial^4 v}{\partial x^4} + \rho A \frac{\partial^2 v}{\partial t^2} = F_0 \sin(\omega t) \quad (N/m)$$

$$(0 \le x \le L) \qquad \text{------} \quad (1)$$

where the applied force per unit length is $F_0 \sin(\omega t)$.

Define the following dimensionless quantities

$$\bar{x} = \frac{x}{L} \qquad \bar{t} = t\sqrt{\frac{EI}{\rho A L^4}} \qquad \bar{v} = \frac{v}{L}$$

$$\beta = \frac{F_0 L^3}{EI} \qquad F_0 = \frac{\beta EI}{L^3}$$

$$\bar{\omega}\bar{t} = \omega t \Rightarrow \bar{\omega} = \frac{\omega t}{\bar{t}} = \omega\sqrt{\frac{\rho A L^4}{EI}}$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \bar{x}}\frac{d\bar{x}}{dx} = \frac{1}{L}\frac{\partial}{\partial \bar{x}} \rightarrow \frac{\partial^4}{\partial x^4} = \frac{1}{L^4}\frac{\partial^4}{\partial \bar{x}^4} \quad \text{..... i)}$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \bar{t}}\frac{d\bar{t}}{dt} = \sqrt{\frac{EI}{\rho A L^4}}\frac{\partial}{\partial \bar{t}} \rightarrow \frac{\partial^2}{\partial t^2} = \frac{EI}{\rho A L^4}\frac{\partial^2}{\partial \bar{t}^2} \quad \text{..... ii)}$$

Subst. in ① gives

$$(EI)\frac{1}{L^4}\frac{\partial^4}{\partial \bar{x}^4}(L\bar{v}) + (\rho A)\frac{EI}{\rho A L^4}\frac{\partial^2}{\partial \bar{t}^2}(L\bar{v}) = \left(\frac{EI}{L^3}\right)\beta \sin(\bar{\omega}\bar{t})$$

$\Rightarrow$

$$\frac{\partial^4 \bar{v}}{\partial \bar{x}^4} + \frac{\partial^2 \bar{v}}{\partial \bar{t}^2} = \beta \sin(\bar{\omega}\bar{t}) \quad \text{--- ②}$$

$$0 \le \bar{x} \le 1$$

In the following, we drop the over-bars for notational convenience. i.e. ② is written as

$$\frac{\partial^4 v}{\partial x^4} + \frac{\partial^2 v}{\partial t^2} = \beta \sin(\omega t) \quad \text{--- ③}$$

where it is understood that v, x, t, ω are dimensionless quantities.

Boundary conditions:

$$\left.\begin{aligned} v(0,t) &= v(1,t) = 0 \\ \frac{\partial^2 v}{\partial x^2}(0,t) &= \frac{\partial^2 v}{\partial x^2}(1,t) = 0 \end{aligned}\right\} \quad \text{--- ④}$$

For boundary conditions ④ the orthonormal modes are

$$X_k(x) = \sqrt{2}\sin(k\pi x) \quad \text{--- ⑤}$$

$$0 \le x \le 1 \quad (k = 1, 2, \ldots)$$

The inner product of X_r and X_s is

$$\langle X_r, X_s \rangle = \int_0^1 X_r X_s \, dx \quad --- \quad (6)$$

and we have

$$\langle X_r, X_s \rangle = \begin{cases} 0 & \text{if } r \neq s \\ 1 & \text{if } r = s \end{cases} \quad --- \quad (7)$$

By the expansion theorem:

$$v(x,t) = \sum_{k=1}^{\infty} X_k(x) \, p_k(t) \quad ---- \quad (8)$$

$$\beta \sin(\omega t) = \sum_{k=1}^{\infty} X_k(x) \, c_k(t) \quad ----- \quad (9)$$

Take the inner product of (9) with X_r and use (7):

$$\langle X_r, \beta \sin(\omega t) \rangle = c_r(t).$$

i.e $$c_r(t) = \int_0^1 X_r \, \beta . \sin(\omega t) \, dx$$

$$= \sqrt{2} \beta \sin(\omega t) \int_0^1 \sin(r \pi x) \, dx$$

$$= -\frac{\sqrt{2} \beta \sin(\omega t)}{r \pi} \Big[\cos(r \pi x) \Big]_0^1$$

$$= \frac{\sqrt{2} \beta}{r \pi} \sin(\omega t) \Big[1 - \cos(r \pi) \Big].$$

$$c_r(t) = \frac{\sqrt{2}\,\beta}{r\pi}\left[1-(-1)^r\right]\sin(\omega t)$$

$$= \frac{\sqrt{2}\,\beta\, a_r}{r\pi}\sin(\omega t) \quad \text{---} \quad (10)$$

where $a_r = 1-(-1)^r = \begin{cases} 0, & r \text{ even} \\ 2, & r \text{ odd.} \end{cases}$

--- (11).

Subst. (8), (9) into (3) and note that

$$\frac{\partial^4 X_k}{\partial x^4} = (k\pi)^4\sqrt{2}\,\sin(k\pi x) = (k\pi)^4 X_k$$

$$\sum_{k=1}^{\infty}(k\pi)^4 X_k P_k + \sum_{k=1}^{\infty} X_k \ddot{P}_k = \sum_{k=1}^{\infty} X_k c_k$$

Take inner product with X_r and use (7):

$$\ddot{P}_r + (r\pi)^4 P_r = c_r \quad \text{---} \quad (12)$$

Using (10):

$$\ddot{P}_r + (r\pi)^4 P_r = \frac{\sqrt{2}\,\beta\, a_r}{r\pi}\sin(\omega t) \quad \text{---} \quad (12)$$

The steady-state solution is of the form

$$P_r(t) = A_r\cos(\omega t) + B_r\sin(\omega t)$$

$$\dot{P}_r = -\omega A_r\sin(\omega t) + \omega B_r\cos(\omega t)$$

$$\ddot{P}_r = -\omega^2 A_r\cos(\omega t) - \omega^2 B_r\sin(\omega t)$$

Subst. in (12):

$$-\omega^2 A_r\cos(\omega t) - \omega^2 B_r\sin(\omega t) + (r\pi)^4\left[A_r\cos(\omega t) + B_r\sin(\omega t)\right] = \frac{\sqrt{2}\,\beta\, a_r}{r\pi}\sin(\omega$$

$$-\omega^2 A_r + (r\pi)^4 A_r = 0 \Rightarrow A_r = 0.$$

$$-\omega^2 B_r + (r\pi)^4 B_r = \frac{\sqrt{2}\,\beta\, a_r}{r\pi}$$

$$\Rightarrow B_r = \frac{\sqrt{2}\,\beta\, a_r}{r\pi\left\{(r\pi)^4 - \omega^2\right\}}$$

$$p_r(t) = \frac{\sqrt{2}\,\beta\, a_r}{r\pi\left\{(r\pi)^4 - \omega^2\right\}} \sin(\omega t) \quad \text{--- (13)}$$

From (8)

$$V(x,t) = \sum_{k=1}^{\infty} \sqrt{2}\,\sin(k\pi x)\,\frac{\sqrt{2}\,\beta\, a_k}{k\pi\left\{(k\pi)^4 - \omega^2\right\}} \cdot \sin(\omega t)$$

$$= \left(\frac{2\beta}{\pi}\right)\sin(\omega t) \sum_{k=1}^{\infty} \frac{a_k \sin(k\pi x)}{k\left\{(k\pi)^4 - \omega^2\right\}}$$

i.e.

$$\bar{V}(\bar{x},\bar{t}) = \frac{2\beta}{\pi}\sin(\bar{\omega}\bar{t}) \sum_{k=1}^{\infty} \frac{a_k \sin(k\pi\bar{x})}{k\left\{(k\pi)^4 - \bar{\omega}^2\right\}} \quad \text{--- (14)}$$

$$\bar{V} = \frac{V}{L}, \quad \bar{\omega}\bar{t} = \omega t, \quad \bar{x} = \frac{x}{L}.$$

Eqn (14) becomes

$$\frac{V(x,t)}{L} = \frac{2\beta}{\pi}\sin(\omega t) \sum_{k=1}^{\infty} \frac{a_k \sin\left(\frac{k\pi x}{L}\right)}{k\left\{(k\pi)^4 - \bar{\omega}^2\right\}} \quad \text{--- (15)}$$

$$V(x,t) = \frac{2F_0L^4}{\pi EI}\sin(\omega t)\sum_{k=1}^{\infty}\frac{a_k \sin\left(\frac{k\pi x}{L}\right)}{k\left\{(k\pi)^4-\bar{\omega}^2\right\}}$$

where $a_k = \begin{cases} 0 & , k \text{ even} \\ 2 & , k \text{ odd.} \end{cases}$ (16)

$$V(x,t) = \frac{2F_0L^4}{\pi EI}\sin(\omega t)\left\{\frac{2\sin\left(\frac{\pi x}{L}\right)}{\pi^4-\bar{\omega}^2} + \frac{2\sin\left(\frac{3\pi x}{L}\right)}{3\left\{(3\pi)^4-\bar{\omega}^2\right\}} + \frac{2\sin\left(\frac{5\pi x}{L}\right)}{5\left\{(5\pi)^4-\bar{\omega}^2\right\}} + \cdots\right\}$$

$$\bar{\omega} = \omega\sqrt{\frac{\rho AL^4}{EI}}$$

Natural Frequencies are

$$\omega_k = k^2\pi^2\sqrt{\frac{EI}{\rho AL^4}}$$

$$(k = 1, 2, \ldots)$$