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AN INTRODUCTION TO ANALYTICAL STRIP TRIANGULATION, WITH A FORTRAN PROGRAM

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PHOTOGRAMMETRIC RESEARCH

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An Introduction to Analytical Strip Triangulation, with a FORTRAN Program

Revised Edition

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An Introduction to Analytical Strip Triangulation with a FORTRAN program

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Introduction

In photogrammetric mapping, an extensive use is made of procedures which serve to increase the number of ground-control points established by field surveying.

For a long time, the most accurate of the available procedures consisted in aerial triangulation of strips of photographs in first-order plotting instruments followed by transformation of the obtained strip coordinates to the required geodetic or map coordinate system.

Analytical aerial triangulation is an alternative to the triangulation in an instrument. It consists in computing the map coordinates of terrain points directly from measurements of the coordinates of their images in the planes of aerial photographs. The orientation of each photograph of a strip at its moment of exposure with respect to the others is computed in a rectangular three-dimensional coordinate system. With this orientation, rays from corresponding images of terrain points in each two successive photographs intersect, and the coordinates of all these points of intersection are computed. Unless the computation is performed in the ground-control system and the conditions for the available ground-control points have been taken into account, the analytical triangulation is also followed by transformation to the ground-control system.

Analytical triangulation has been considered a potential method since the beginning of photogrammetric mapping. However, before the advent of the electronic computer, the required computations were too time-consuming. The electronic computer has made analytical triangulation a practical possibility.

Analytical aerial triangulation has a number of advantages over triangulation on first-order plotting instruments.

A greater accuracy can be achieved because the measured coordinates can be corrected for all determinable errors in the position of the photographic image. Film distortion can be taken into account when a grid plate is used in front of the negative or, to a lesser degree, when the camera is provided with a sufficient number of fiducial marks. Lens distortion can be compensated, limited only by the accuracy with which the camera has been calibrated and its stability. In instrumental triangulation this is not the case, or at least not to this extent: for example, no corrections can be given for irregular film distortion and for asymmetric lens distortion. Corrections can be applied also for distortion caused by atmospheric refraction. If desired, corrections can be given to eliminate curvature of a strip due to curvature of the earth.

A greater accuracy will be achieved also because analytical triangulation is not restricted by some of the limitations of instrumental triangulation. Triangulation instruments, however accurate, always have their imperfections. The bundle of rays, defined by the image points in a photograph, is not reproduced with mathematical precision. The model of the terrain obtained from a strip of photographs will therefore always be more or less distorted.

In analytical triangulation the bundles of rays are defined by mathematical formulas. The accuracy of the computations is limited only by the number of decimal places used. The only instrumental errors that occur are those in the reading of coordinates on the stereocomparator. Since a precise stereocomparator is a much simpler machine than a first-order plotting instrument, the sources of instrumental errors are fewer in number and their effect can be kept much smaller.

Another limitation of the first-order plotting instruments is in the accuracy with which relative orientation can be established. An approximate orientation is established first. It is then adjusted, either empirically or by a numerical procedure. With the empirical procedure, the result depends more or less on the preference of the operator as to the extent to which he should go in reducing the parallaxes and as to which orientation elements to use. With the numerical procedure, the corrections to the orientation elements are computed unambiguously from observed parallaxes. Application of these corrections to the instrument readings, however, does not generally bring about the expected change in the parallaxes. This is caused by the fact that corrections to the orientation elements cannot be made with mathematical precision. As a result, small but perceptible systematic parallaxes are often left. In analytical triangulation this is not the case: the relative orientation as defined by observed coordinates can be established with any required degree of accuracy.

Furthermore, in instrumental triangulation, only a limited accuracy is reached in the centering of the photograph in the plate holder. This results in a distortion of the bundle of rays. In analytical triangulation, the centering is computed from the readings of the fiducial marks and does not depend upon the positioning of the photograph in the plate holder.

Analytical triangulation also promises economical advantages. A higher accuracy will make it possible to reduce the number of ground-control points, and thus the cost of field surveys, without reducing the accuracy of the produced maps. The cost of the photogrammetric equipment can also be lower. A precise monocomparator is a much simpler instrument than a first-order plotter and is much less expensive. At the present time, the price of a precise stereocomparator is very high but eventually it will have to come down to a level which reflects the relative simplicity of this instrument.

The above advantages make analytical triangulation an attractive proposition. Accordingly, as early as 1953, a method of analytical triangulation was developed at the Photogrammetric Research Section of the Division of Applied Physics of the National Research Council of Canada.

This method was first programmed for the Ferut electronic computer at the University of Toronto, one of the very few electronic computers in Canada at that time [1]. Subsequently, the method was programmed for the IBM 650 and the program was adapted for use on the IBM 1620 [2]. Card decks of this program have been supplied to the research organizations and mapping agencies who requested this.

When the IBM 1620 at the NRC laboratories was replaced by an IBM S/360, the method was reprogrammed in FORTRAN. This program is described in the 1966 edition of this publication. It has been in use since then with only very minor modifications.

The increasing use of analytical triangulation, shown by the present production of comparators and the demand for the program, has now made it worthwhile to revise the program more thoroughly. The main purposes of the revision have been to facilitate the use of the program in mass production and to make possible the division of the program into subroutines, if that should be needed. Considerable changes have been made also in the operating instructions and in the input and output formats and, therefore, one should use either only the new program or only the old one.

As in the earlier FORTRAN version, where possible, only the more basic FORTRAN statements are used. For instance, no use has been made of logical IF statements, of mixed-mode arithmetic, and of disk and tape statements. This serves to make the program usable, with the modification of only the read and write statements, with various compilers.

The first chapter of this publication gives a short analysis of different analytical triangulation procedures with special emphasis on the procedure used in the FORTRAN program. More elaborate analyses can be found in references [3] and [4]. The following four chapters treat the mathematical formulation while the last chapter contains a description of the FORTRAN IV program, operating instructions, and a listing of the statements.

I. Discussion of procedures

1. Photograph coordinates

The process of analytical triangulation starts with the reading of the coordinates of corresponding image points on a comparator.

These readings are first converted to photograph coordinates with origin in the principal point. The photograph coordinates are corrected for the effects of film distortion, lens distortion, and refraction. Corrections for earth curvature may be given if the strip coordinates are to be directly transformed to or produced in the map coordinate system, without some geodetic system as an intermediate link.

2. Orientation procedure

From the corrected photograph coordinates, the map coordinates of the required points must be computed. This computation is most conveniently performed using a spatial rectangular coordinate system. In this system, the absolute orientation of each photograph is determined. Subsequently, the spatial coordinates of all measured points are determined by intersecting corresponding rays. If the spatial coordinate system is not identical with the map coordinate system, this computation is followed by transformation of the obtained coordinates to that system.

The absolute orientation of a photograph is expressed by six elements, as for instance the three rectangular coordinates of its projection centre and three parameters that define the position of the photograph axes with respect to the axes of the coordinate system.

These six elements must be determined from the conditions which the projecting rays through corresponding image points must fulfill.

There are three types of conditions:

i. The projecting rays from corresponding image points in two consecutive photographs must intersect.

ii. The rays from corresponding image points in three consecutive photographs must intersect at one point. This condition can be expressed by specifying that two pairs of corresponding rays must intersect at the same distance below the common photograph.

iii. The given coordinates of a ground-control point must satisfy the equations of the rays from its image points.

The conditions of the first two types express relations which must exist between the orientation elements of adjoining photographs. They determine primarily the relative orientation of these photographs.

As a result, the absolute orientation of each photograph cannot be determined independently of that of the others. It can be determined only by one of the two following procedures: either successively for one photograph after the other or for all photographs simultaneously.

The first of these procedures resembles the triangulation in a plotting instrument. Independently for each strip that is triangulated, it produces coordinates in a three-dimensional coordinate system which is not related to the ground-control system. This strip triangulation procedure is the subject of this publication.

This procedure breaks the computation up into small steps. An arbitrary orientation of the first photograph is assumed. The orientation of each following photograph is then computed in succession. Here, the procedure used in instrumental triangulation is followed: the orientation consists in the relative orientation of each photograph with respect to the preceding one followed by scaling of the resulting model. The relative orientation can be established by making five pairs of corresponding rays intersect. The resulting model can then be scaled to the preceding one by making one height or one distance in the two models equal. Generally, more points will be measured in each model than the minimum that is necessary to establish the relative orientation and the scale. This will be done partly as a check on errors and partly to increase the accuracy. The relative orientation and the scale are then adjusted separately.

As an alternative in this procedure, the six orientation elements of a photograph could be computed simultaneously. For this computation, six independent condition equations are necessary [5,6]. One possibility would be the measurement of five pairs of corresponding points, specifying that four of the pairs of corresponding rays need merely intersect while the fifth must intersect in the point established in the preceding model. A second possibility would be the measurement of four pairs of corresponding points, specifying that two pairs of corresponding rays need merely intersect while the other two must intersect in points established in the preceding model. If more measurements are available, an adjustment could be carried out for all six elements simultaneously using all available pairs of corresponding points and all available points from the preceding model.

This alternative has a disadvantage. Errors in the orientation of a photograph result in model deformation, especially in height. If this occurs, and two or more well separated points in such a model are used in the adjustment of the next model, that model will be deformed accordingly, causing errors in all orientation elements of its second photograph. As a result, deformation of one model causes deformation of all following models in succession. Consequently, each model is affected by deformation of all preceding models, but not by deformation of any following model. Therefore, the result of the triangulation depends on the choice of the model used to start the triangulation, that is, in practice, upon the direction of the triangulation. The disadvantage is sufficient to reject this alternative.

The second of the above procedures consists in the computation of the elements of absolute orientation of all photographs simultaneously, using all available condition equations for intersecting corresponding rays and, if one wishes, all those for ground-control points [7,8]. This involves the simultaneous solution of as many equations as there are elements of absolute orientation, i.e., six times the number of photographs.

A redundant number of measurements should be available and a rigorous method of adjustment such as the method of least squares should be applied. In the method of least squares, the condition equations serve for the formation of normal equations. Since the condition equations express only relations between the orientation elements of adjoining photographs, the non-zero elements in the matrix of the normal equations are contained in a band along the diagonal. The required storage space and computation time are roughly proportional to the number of models. The formation and solution of the equations poses no serious problem.

If the simultaneous procedure is used, the condition that three corresponding rays must intersect in one point does not give rise to the objectionable one-directional error propagation. However, the computations require a multiple of the data storage and computation time required by the strip triangulation procedure.

The first three computers used by the Photogrammetric Research Section made it necessary to use the strip triangulation procedure. Ferut, the first computer, had sufficient storage space for the simultaneous computation, but could not operate for the required length of time without breakdowns. Even using the step-by-step procedure, breakdowns occurred and parts of the triangulations had to be repeated to cross gaps. The IBM 650, which was the second computer, had at first only 2000 words. This computer, with a 4000-word drum, and the IBM 1620 with 40000 decimal digits, which was subsequently used, had sufficient storage space for the normal equations of the simultaneous solution, but not for all data. The required iterative solution and the subsequent intersecting of rays would have necessitated reading in the data several times. This would have been a somewhat cumbersome procedure and the computation time would have been rather long.

The FORTRAN program described in this publication is also based upon the strip triangulation procedure. It has been developed from a FORTRAN II version which was first prepared for the IBM 1620 with 40000 digits and floating-point hardware.

3. Iterative solution of the condition equations

Analytical formulation of the conditions of intersection produces condition equations that are non-linear in the orientation elements. To solve these equations, they must be differentiated and the resulting linear equations must be used for the determination of the orientation elements in an iterative procedure. Two procedures are possible:

i. The assumed approximations of the orientation elements which are substituted into the differential equations are the same for each iteration. Since in the case of strip triangulation the orientation elements can be chosen in such a way that they are small quantities, the value zero may be chosen as the approximation for each.

When this procedure is used, the points for relative orientation are often chosen in fixed positions in a regular pattern [9,10]. The coefficients and the solution of the equations can then be computed in advance using desk computers. This gives the corrections to the approximate values of the orientation elements as linear functions of the want of intersection in the measured points. These functions are used for all models of a strip. The advantage of this method is the small amount of computation required per iteration. Its disadvantages are the large number of iterations that is required if the assumed approximations of the orientation elements differ much from the correct values and the restriction that is imposed upon the position of the orientation points.

It is possible to use this procedure without imposing this restriction [22]. In that case, the electronic computer must compute and presolve the linear equations once for each model. Especially in the case of incomplete models and of relief, where the points cannot be chosen in a regular pattern, this will improve the convergence of the iterative procedure.

ii. Alternatively, the coefficients can be computed for each iteration using the latest approximate values of the unknowns and the actual positions of the points in the photographs. In this case, the electronic computer must compute and solve the linear equations for each iteration. Because the coefficients are valid not only for the actual positions of the points but also for the latest approximate values of the unknowns, this procedure requires the smallest number of iterations and it converges even if the assumed approximations differ very much from the correct values. If speed or storage space are a problem, an approximate orientation on five points can be performed first, followed by an adjustment using all points.

Because of these advantages, the second procedure has been used in the FORTRAN program. With differences in tilts of the photographs of less than two degrees, two iterations have proved to be sufficient. Even with a convergence of the photograph axes of 45°, the FORTRAN program requires only three iterations. In both these cases, the approximation used in the first iteration consists in the assumption of parallel axes.

4. Adjustment of the relative orientation

In the FORTRAN program, the relative orientation is based directly upon the condition of intersection of corresponding rays. This is only a matter of convenience: it involves less computation than basing it on the condition that the Y-parallax or the shortest distance between corresponding points must be equal to zero [3].

The adjustment of the relative orientation is performed with the method of least squares. It is based upon the requirement that the sum of the squares of the corrections to the photograph coordinates which make the rays intersect must be a minimum.

The formulas have been developed for the case of unequal accuracy and correlation between the photograph coordinates. In practice, equal accuracy and freedom from correlation will often be assumed either for the sake of simplicity or because no reliable values are available. The FORTRAN program contains the option of either assuming equal accuracy and freedom from correlation or using an experimental formula derived for a wide angle camera with a 6" focal length.

5. Intersection of rays

After the orientation of each photograph, the projecting rays from two corresponding points will intersect only if the corrections to the photograph coordinates which follow from the adjustment of the relative orientation are actually applied.

In the NRC programs, these corrections are not computed. Instead, in the earlier programs, the procedure in instrumental triangulation has been followed: the strip is triangulated roughly in the X-direction. The X- and Z-coordinates of a point are defined as being equal to those of the points on the corresponding rays at the height where their X-parallax is equal to zero. The Y-coordinate is computed as the mean of the Y-coordinates of those points, the Y-parallax as the difference. For vertical photographs and on the assumption of equal accuracy of the coordinate readings and freedom from correlation, this procedure gives the same point as is obtained after correction of the photograph coordinates.

In the FORTRAN program, the point of intersection is defined as the point midway between the rays on their line of shortest distance. The want of correspondence is defined as their shortest distance.

It can be a matter of opinion which point is the best: the least-squares point, the Y-parallax point or the line-of-shortest-distance point. If the want of correspondence is smaller than 10μ at photograph scale, the difference is negligible, except perhaps for points in the corners of super-wide-angle photographs. If the want of correspondence is considerably larger, either the point or the whole triangulation is not of a good quality and the question of best choice can only be of academic interest. The first and the third definition have the advantage that the obtained point is independent of the orientation of the strip.

6. Transformation and adjustment

If, as with the FORTRAN program, the strip coordinates are computed with respect to a preliminary coordinate system, these coordinates must subsequently be transformed to map coordinates and terrain heights. This can be done either via coordinate systems on the earth or directly. The direct way is the simpler one. A rectangular three-dimensional coordinate system is then assumed, having as coordinates the two coordinates of the map projection system and the terrain heights.

A separate program has been written for this transformation and adjustment of triangulated strips. It is described in reference [11].

II. Photograph coordinates and corrections

1. Conversion from comparator measurements to photograph coordinates

Various stereocomparators and monocomparators are now available for performing the measurements needed in analytical triangulation.

In most of these instruments, the position of an image point in the plane of a photograph can be measured with respect to a rectangular coordinate system which has a sufficient range to cover the whole photograph. Usually, the origin of this coordinate system is outside the photograph.

The analytical triangulation requires coordinates with the origin in the principal point. These photograph coordinates are obtained by subtracting the instrument coordinates of the principal point from those of the image points.

Usually, the principal point is not marked on the photograph and, consequently, is not measured. It is then necessary to measure the fiducial marks and to compute each coordinate of the fiducial centre as the mean of the corresponding coordinates of the fiducial marks. The coordinates of the principal point are then derived from those of the fiducial centre with the help of the calibration data of the camera. For practical purposes, these two points can usually be considered to be identical.

With some stereocomparators, for the right photograph parallaxes are read instead of coordinates. Such stereocomparators can be used for the measurement of vertical aerial photographs. The parallaxes are read on short screws which can be made more accurate than a screw which covers the whole range of a photograph. The parallaxes must be converted to instrument coordinates by adding them to or subtracting them from the coordinates read simultaneously for the left photograph.

Some comparators are equipped with a large number of measuring marks in the pattern of a rectangular grid. Each image point is then measured with the nearest measuring mark. At the NRC laboratories, for instance, a monocomparator has been developed which is equipped with 12 × 12 measuring marks placed at 20 mm intervals. Such comparators have measuring screws which are not much longer than the distance between adjacent marks. Instrument coordinates are here obtained by adding the measurements made with the screws to the calibrated coordinates of the used measuring mark.

At NRC, the computation of the instrument coordinates of the principal points from the measurements of the fiducial marks is performed with a desk calculator. The conversion from instrument coordinates to photograph coordinates is included in the triangulation program.

The measurements made with the Zeiss Jena stereocomparator at NRC require the conversion from parallaxes to instrument coordinates. The NRC monocomparator measurements must be corrected for deviations in the position of the measuring marks from an ideal 20 mm grid.

These two computations are also performed by the computer. They require small additions to the regular program which have not been included in the listing of the FORTRAN statements.

2. Corrections for film distortion

A few photogrammetric cameras are equipped with a register glass with a grid in the focal plane. This grid is therefore reproduced on the negative. For each image point, a pointing can now be made at the point itself and at one or more of the nearest grid intersections. Subtraction of the coordinate readings for image point and intersection gives the coordinates of the image point, with a grid intersection as origin. These coordinates are added to the calibrated coordinates of the grid intersection and the resulting coordinates are treated as instrument coordinates. In this way, the effect of film distortion on these coordinates is largely eliminated.

In the absence of a register glass with grid, measurement of the fiducial marks can give an indication of the distortion. Unfortunately, most cameras have only four fiducial marks. That is not sufficient for a reliable determination of film distortion over the whole area of the photograph. Still, the measurements can be used to eliminate at least part of the systematic distortion.

At NRC, measurement of the distances of fiducial marks is used to determine average values of the change of scale of the photographs of a strip in the direction of the photograph coordinates. These average values are used to determine correction factors which the FORTRAN program can apply to the photograph coordinates.

3. Lens distortion

The distortion of a bundle of rays by a lens causes displacement of the images of the measured points with respect to their ideal positions. The latter follow from an assumed position of an undistorted bundle in the image space. The displacement of the images is called the lens distortion. It can have radial and tangential components.

The calibration of a camera is the procedure of determining the lens distortion, the position of the principal point, and the focal length. Knowledge of these quantities makes it possible to construct an undistorted bundle in the image space.

The principal point can be defined in various ways. At NRC, the principal point of autocollimation is used. This is the point where a ray which in the object space is perpendicular to the plane of the photograph intersects that plane.

The centre of the perspective bundle of rays in the image space is placed on the perpendicular in the assumed principal point, and at a distance equal to the calibrated focal length f from the plane of the photograph.

Let α be the angle which a ray in the object space makes with the perpendicular and let r be the distance from the principal point to the point where the ray in the image space intersects the photograph. The radial distortion Δr is the radial distance between the actual and the ideal point of intersection, and therefore:

$$\Delta r = r - f \tan \alpha \qquad (3.1)$$

This equation shows clearly that the radial distortion is a function of the assumed value of the focal length. At NRC, that value of the focal length is determined which makes the maximum difference between the values of the radial distortion and standard reference values for the lens as small as possible. For cameras for which reference values are not available, that value of the focal length is determined which makes the maximum value of the distortion as small as possible. That value of the focal length is called the calibrated focal length.

Finally, the orientation of the bundle in the image space can be further fixed by specifying that in the principal point the lens distortion is equal to zero. Using the principal point of autocollimation this means that the ray which is in the object space perpendicular to the plane of the photograph is in the image space also perpendicular to this plane and intersects it in the principal point. Equation (3.1) can now be used to compute the radial distortion for rays of known angles α from measurements of the radial distances r.

With the recently installed camera calibrator at the NRC laboratories, the radial distortion along the four half-diagonals to the corners of the photograph can be determined at angular distances of 2 13/16° and multiples of this till 59 1/16° from the principal point. Occasionally, the radial distortion will be determined also along the four half-diagonals to the middle of the sides. The tangential distortion is considerably smaller than the radial distortion and is usually not determined.

The distortion caused by a manufactured lens is in practice not the same as the theoretical distortion inherent in the lens design. This is a result of a small decentering of the lens components and of other manufacturing defects. The difference is roughly equal to the effect of adding a small prism to the lens. As a result, the principal point of autocollimation does not coincide with the point where its defining ray in the object space intersects the plane of the photograph. Also, the distortion is asymmetric with respect to the principal point of autocollimation, and tangential distortion occurs.

The distortion can be referred to a different principal point. This implies that the centre of the perspective bundle in the image space is shifted to the perpendicular in the new principal point and that the bundle is rotated by the amount which makes rays to points in the area of the principal points continue to intersect the plane of the photograph in the same points. As a result of the shift and the rotation of the bundle, the pattern of the distortion in the plane of the photograph will change.

Let the shift and the rotation be made in such a way that the asymmetries in the radial distortion become as small as possible.

The new principal point is then called the principal point of best symmetry.

If a shift of the perspective centre parallel to a diagonal is called a, the associated rotation - a/f, and a shift perpendicular to the plane of the photograph b, as shown in Figure 1, the resulting radial shift of the point of intersection of a ray and the diagonal is

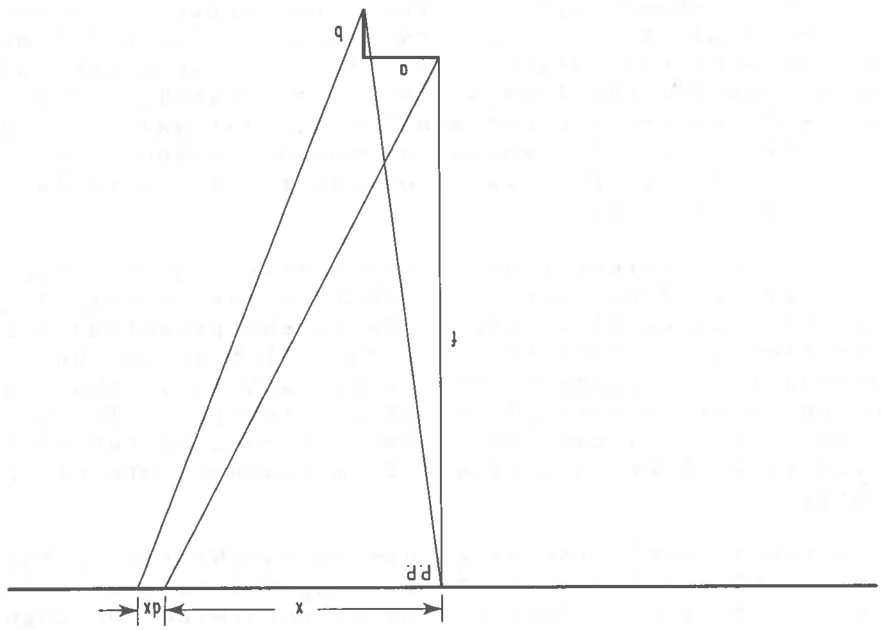

Figure 1. The effect of changes in the choice of principal point and calibrated focal length upon the radial distortion.

$$dx = a - \frac{x^2 + f^2}{f^2} a + \frac{x}{f} b \qquad (3.2)$$

Each of the three parameters contributes one term to this equation.

The radial shift dx, the distance x, and the parameters a and b each have a positive direction. The positive directions have been chosen in such a way that the values in Figure 1 are all positive. On the other hand, the radial distance r is always positive and the radial distortion is positive when it is directed away from the principal point.

One can specify that for two points on opposite sides of the principal point and at equal distances from it the shifts dx must cancel the radial distortions valid for the principal point of autocollimation. If the radial distortions at $x_1 = +r$ and at $x_2 = -r$ are called Δr_1 and Δr_2, respectively, the required radial shifts are $dx_1 = -\Delta r_1$ and $dx_2 = +\Delta r_2$. Substituted into equation (3.2), this produces two equations from which the required shifts a and b can be computed. The solution is:

$$2\,\frac{r^2}{f^2}\,a = \Delta r_1 - \Delta r_2$$

$$-2\,\frac{r}{f}\,b = \Delta r_1 + \Delta r_2 \qquad (3.3)$$

The shifts a and b can thus be computed for each pair of radial distortions. Usually they will be different for each pair and, therefore, adjusted values should be computed from the equations (3.3) by the method of least squares. This gives

$$a = \frac{\Sigma\left\{2\,\frac{r^2}{f^2}\,(\Delta r_1 - \Delta r_2)\right\}}{\Sigma\left\{2\,\frac{r^2}{f^2}\right\}^2} \qquad (3.4)$$

If a calibrated focal length has already been computed, the shift b is not needed.

The computation of the shift a can be performed for the two main diagonals. This gives the position of the principal point of best symmetry with respect to the previously used principal point. Table 1 gives an example of this computation applied to a 6" Hilger and Watts F.105 camera with a Wild Aviogon lens and register glass.

Table 1. An example of the effect of the choice of principal point upon the radial lens distortion.

Angle	Radial distortion along four half-diagonals									
	with p.p. of autocollimation					with p.p. of best symmetry				
	NW	SW	SE	NE	Mean	NW	SW	SE	NE	Mean
10°	+ 2μ	+ 4μ	+ 4μ	+ 3μ	+ 3μ	+ 2μ	+ 4μ	+ 4μ	+ 3μ	+ 3μ
20°	- 1	+ 4	+ 5	+ 4	+ 3	+ 1	+ 5	+ 3	+ 3	+ 3
30°	-14	- 6	- 5	- 4	- 7	- 5	- 4	-14	- 6	- 7
40°	-12	0	+12	+15	+ 4	- 2	+ 4	+ 2	+11	+ 4
45°	+18	+17	+42	+24	+25	+32	+23	+28	+18	+25
Distance from fiducial centre to p.p. of autocollimation: less then 10μ. Shift to p.p. of best symmetry: NW-SE 14μ, NE-SW 6μ.										

It should be noted that this computation is based upon the assumption of an error-free measurement of an image produced at the principal point. If one does not wish to assume this, the rotation of the perspective bundle should not be rigidly connected with the shift a. This leads to an equation for each of the measured points in which the shift and the rotation occur as independent unknowns.

The FORTRAN program contains provision for correction of the photograph coordinates for symmetrical radial lens distortion. At NRC, this distortion is taken to be the mean of the radial distortions along the half-diagonals to the four corners. Usually, the fiducial centre is accepted as the principal point.

As Table 1 shows, the mean distortion tends to be independent of the choice of principal point. For a good camera and with the choice of the principal point of autocollimation, the distortions along each half-diagonal differ only a few microns from the mean. This justifies the use of the above procedure. However, where the greatest possible accuracy is needed, it will be advisable to apply corrections for asymmetrical radial distortion and for tangential distortion.

Values of the radial distortion at distances from the principal point where the camera calibration does not provide them can be found by plotting the available values against the radial distance and drawing a smooth curve through the plotted points. The standard distortion curve should be used as a guide, especially at the ends of the diagonals where sufficient calibration data are often not available.

It is often difficult to estimate the best position of some parts of the curve with an accuracy of one or two microns. This causes a small amount of arbitrariness which can be avoided as follows.

Regard either the deviations of the distortion from the standard values for the lens or the distortions themselves at equal intervals along a half-diagonal as unknowns which shall be computed. Take the intervals so small that within each interval the distortion can be treated as a linear function of the position. Formulate the condition equations which specify that at the points where the camera calibration has provided values for the distortion, the distortion should have these values. Formulate also condition equations which specify that the computed distortion or its deviation from the standard values should vary in a smooth manner. This can be done, for instance, by specifying that the values of each three successive unknowns, plotted against the radial distance to the principal point, should lie on a straight line. This procedure leads to more condition equations than there are unknowns. They can be solved by the method of least squares.

4. Photogrammetric refraction

4.1 Standard atmosphere and flat earth

On their paths from the terrain to the camera, the light rays pass through air of decreasing density. As a result, they are refracted away from the vertical.

As shown in Figure 2, left, this refraction causes a small angle at the camera between the ray from a terrain point and the straight line from this point. The straight line makes a smaller angle with the vertical through the perspective centre of the camera than the ray makes at this centre. Consequently, the refraction causes a displacement of the photographic image away from the nadir point in the photograph.

This angle at the camera is called the photogrammetric refraction. It is a function of the refractive index of the air in all the points along the ray. The refractive index is a function of temperature, pressure, humidity and CO_2-content or, in short, of the density of the air.

Since these quantities cannot be measured along a whole ray, it is convenient to assume that the photogrammetric refraction in the actual atmosphere is the same as that in one of the present

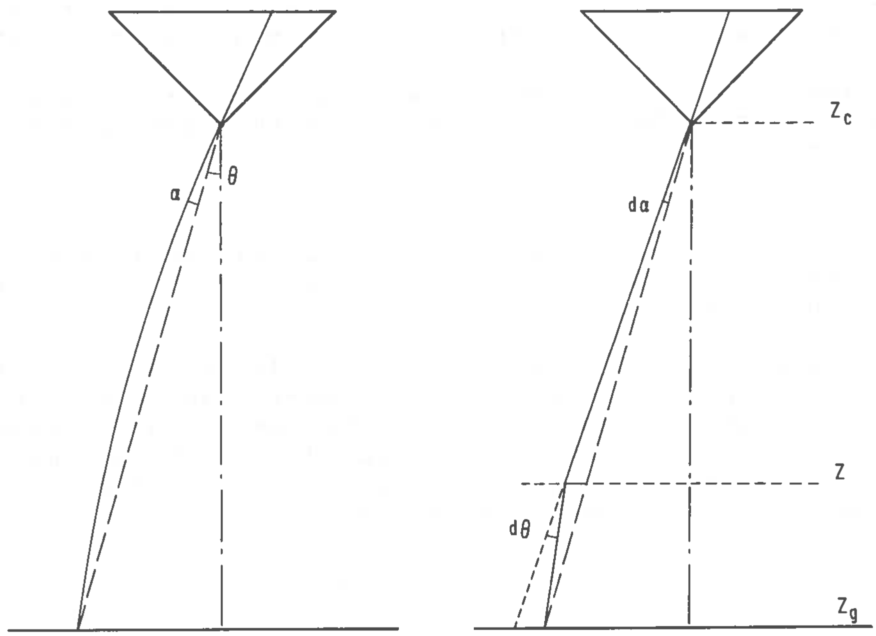

Figure 2. Refraction in the atmosphere and at a boundary between layers of different density.

standard atmospheres. These are:

i. The ICAO Standard Atmosphere, 1952 of the International Civil Aviation Organization [12].

ii. The ARDC Model Atmosphere, 1959 of the Air Research and Development Command of the U.S. Air Force [13].

iii. The U.S. Standard Atmosphere, 1962 of the U.S. Committee on Extension to the Standard Atmosphere [14].

Up to 20 km, these three atmospheres are practically the same. The latter two extend beyond 20 km, and up to 32 km the difference in their densities increases with increasing height to 3.5%.

Bertram [15] gives a simple method for computing the photogrammetric refraction in a standard atmosphere, using a table for the density. This method will here be followed in principle. However, the necessary formulas will be derived from Snell's well-known law of refraction instead of from the velocity of wavefronts, as Bertram does. Further, the table for the refraction will be computed in a simpler way and will give values of the refraction at all multiples of the lowest flying height instead of at odd multiples only. Finally, Bertram's formula for computing the refraction when the ground level is above sea level will be replaced because it leads to gross errors.

For the computation of the refraction, the atmosphere may be assumed to consist of a series of thin concentric shells, each of constant density. These densities decrease with increasing height of the shells. In this model of the atmosphere, refraction is caused by the changes in density at the boundaries of the shells.

According to Snell's law refraction, for a light ray which pierces a boundary, the product n sin θ is the same on both sides of the boundary:

$$n \sin \theta = \text{constant} \qquad (4.1)$$

Here, n is the index of refraction and θ is the angle between the light ray and the perpendicular to the boundary at the point where it is pierced by the ray.

The angle of refraction, that is the difference dθ between the values of θ on the two sides of the boundary, can be expressed as a function of the difference dn between the indices of refraction by differentiating equation (4.1). Disregarding a minus sign obtained during the differentiation, this gives the following relation between the absolute values of dθ and dn:

$$d\theta = \frac{dn}{n} \tan \theta \ , \qquad (4.2)$$

where dθ is expressed in radians.

It follows from Figure 2, right, that each refraction dθ contributes to the photogrammetric refraction the amount

$$d\alpha = \frac{Z - Z_g}{Z_c - Z_g} \, d\theta \qquad (4.3)$$

where Z, Z_g, and Z_c are the height of the boundary, of the ground, and of the camera, respectively. The photogrammetric refraction is the sum of the angles dα over all boundaries between the ground height and the height of the camera.

Since the tables for the standard atmospheres give the density but not the refractive index as a function of the altitude, it is preferable to write dθ as a function of the change in density rather than as a function of the change in refractive index.

According to textbooks on meteorological optics, the relation between density and refractive index has been determined by experiment and has been expressed in various formulas. One of the simplest is

$$n^2 = 1 + 2c\rho \qquad (4.4)$$

where ρ is the density in, for instance, kg/m^3.

The constant c in this equation, and therefore the index of refraction, is a function of the wavelength of the light. According to Edlén [16], the formula which over the whole range of the visible spectrum agrees best with the results of experiments is for standard air:

$$(n-1)10^7 = 643.28 + \frac{294981.0}{146 - 1/\lambda^2} + \frac{2554.0}{41 - 1/\lambda^2} \qquad (4.5)$$

where the wavelength λ is measured in microns. This standard air is at 15°C, at normal pressure (760 mm Hg at 0°C), has 0.03% CO_2 (by volume at 0°C), and is dry. It has practically the same composition as the air in the standard atmospheres and the same density of 1.2250 kg/m^3 at sea level as the standard atmospheres.

If the values of the index of refraction, derived from equation (4.5) for different wavelengths, and the above value of the density are substituted into equation (4.4), the following values of the constant c are obtained.

For $\lambda = 0.42\ \mu$, $c = 0.00023004$

For $\lambda = 0.56\ \mu$, $c = 0.00022667$

For $\lambda = 0.66\ \mu$, $c = 0.00022550$

The first and the last of these values of λ are near the ends of the effective range of panchromatic film; the second one is a suitable average.

Differentiation of equation (4.4) gives

$$\frac{dn}{n} = \frac{c}{n^2}\, d\rho \qquad (4.6)$$

and, since from ground level to empty space n varies from about 1.00022 to 1, over this range and for $\lambda = 0.56$ microns

$$\frac{dn}{n} = 0.0002266\, d\rho \qquad (4.7)$$

Kaye and Laby [17] use the simplified formula $n - 1 = c\rho$. From their values of $n - 1$ and ρ, reduced to the temperature of 15°C and the wavelength of 0.56 microns, one finds $c = 0.0002261$. Leyonhufvud [18] employs a coefficient of 0.22607 for the D-line. Reduced to the same values, this gives $c = 0.0002265$.

Considering the uncertainty in the fourth significant digit of c and the limited number of digits that is needed, that digit may be omitted.

Combining now equations (4.3), (4.2), and (4.7), and forming the sum over all boundaries between ground level and camera height, one obtains for the photogrammetric refraction the expression

$$\alpha = 0.000226\, \frac{\tan\theta}{Z_c - Z_g}\, \Sigma((Z-Z_g)d\rho) \qquad (4.8)$$

In this equation, the factors which are the same for all $d\alpha$ have been placed outside the summation. The equation is identical with the one derived by Bertram.

Equation (4.8) makes the computation of the photogrammetric refraction in one of the standard atmospheres very simple. For these atmospheres, the density is listed at discrete values of the geometric height above sea level. In the above model of the atmosphere, the boundary between two shells of constant density is now chosen midway between two heights for which the density is listed and the densities of the two shells are taken to be those listed densities. In this way, each boundary produces a contribution $(Z-Z_g)d\rho$ to the sum in equation (4.8), Z being the height of the boundary and $d\rho$ being the difference between the densities at the two table heights. For each flying height for which the photogrammetric refraction is required, the contributions from all boundaries between it and ground level are added and subsequently the sum is multiplied by the factors which have been placed outside the summation.

Table 2 gives the result of this computation for a ray that makes an angle of 45° with the vertical, assuming flying heights above sea level of up to 32 km and four different ground heights. To obtain the best possible computational accuracy, densities at intervals of less than 500 meters must be used especially for the lower shells. Actually, the densities at all multiples of 100 meters from sea level to 20000 m and at all multiples of 200 meters from 20000 m to 32000 m have been used. This places the boundaries between the shells at 50 m, 150 m, 250 m, etc. The table values are accurate to within one digit of the least significant digit.

Values of the refraction for unlisted flying heights can be computed with sufficient accuracy by linear interpolation in the column of the required ground height. Values for unlisted ground heights can be computed by interpolation between values for listed ground heights. For accurate values, second differences must be used in the interpolation and around the flying height of 11000 m, where a discontinuity occurs, the interpolation must be made between values of the same flying height above sea level, not between values of the same flying height above ground.

It follows from equation (4.8) that for angles θ other than 45° the values of the refraction can be obtained by multiplying the table values by tan θ.

The table gives the values for the U.S. Standard Atmosphere, 1962. The values for the ARDC Model Atmosphere, 1959 are the same, except for flying heights from 21 to 25 km where they are 0.1 μrad smaller and for flying heights from 27 to 32 km where they are 0.1 μrad larger.

In the preceding, the table values for ground heights above sea level have been computed in the same way as those for the ground height at sea level. However, it is possible to compute them directly from the latter. For this purpose, the sum in equation (4.8) is written

$$\sum_s^c Z d\rho - \sum_s^g Z d\rho - Z_g \sum_g^c d\rho \ .$$

Table 2. Photogrammetric refraction for a ray at 45° with the vertical in the U.S. Standard Atmosphere, 1962, in microradians.

Flying height above sea level	Photogrammetric refraction for ground heights of 0.0 km	1.0 km	2.0 km	4.0 km
0.5 km	6.5			
1.0	12.6	0.0		
1.5	18.5	6.0		
2.0	24.1	11.7	0.0	
2.5	29.3	17.1	5.6	
3.0	34.3	22.3	10.9	
3.5	39.0	27.1	15.9	
4.0	43.5	31.7	20.6	0.0
4.5	47.7	36.1	25.1	4.7
5.0	51.6	40.2	29.3	9.2
5.5	55.3	44.0	33.3	13.5
6.0	58.8	47.6	37.0	17.5
6.5	62.1	51.0	40.6	21.3
7.0	65.1	54.2	43.9	24.8
7.5	67.9	57.2	47.0	28.2
8.0	70.6	59.9	49.8	31.3
8.5	73.0	62.5	52.5	34.2
9.0	75.2	64.9	55.0	37.0
9.5	77.3	67.1	57.4	39.5
10.0	79.2	69.1	59.5	41.9
10.5	80.9	70.9	61.5	44.1
11.0	82.5	72.6	63.3	46.1
11.5	85.0	75.2	66.0	49.0
12.0	87.1	77.4	68.3	51.5
12.5	88.8	79.3	70.2	53.7
13.0	90.2	80.8	71.8	55.5
13.5 km	91.3	82.0	73.2	57.0
14.0	92.2	83.0	74.2	58.2
14.5	92.8	83.7	75.1	59.2
15.0	93.3	84.2	75.7	60.1
15.5	93.5	84.6	76.2	60.7
16.0	93.6	84.8	76.5	61.2
16.5	93.6	84.9	76.6	61.5
17.0	93.4	84.8	76.6	61.7
17.5	93.2	84.6	76.6	61.8
18.0	92.8	84.3	76.4	61.8
18.5	92.3	84.0	76.1	61.7
19.0	91.8	83.5	75.7	61.5
19.5	91.2	83.0	75.3	61.3
20.0	90.5	82.4	74.8	61.0
21.0	89.1	81.2	73.8	60.3
22.0	87.5	79.8	72.6	59.4
23.0	85.8	78.3	71.2	58.4
24.0	84.0	76.7	69.8	57.2
25.0	82.2	75.0	68.2	56.0
26.0	80.3	73.4	66.7	54.8
27.0	78.4	71.6	65.1	53.5
28.0	76.6	69.8	63.6	52.2
29.0	74.7	68.2	62.0	50.9
30.0	72.9	66.5	60.5	49.6
31.0	71.1	64.8	59.0	48.4
32.0	69.4	63.2	57.5	47.1

Refraction in the tentative region

Flying height above sea level	0.0 km	1.0 km	2.0 km	4.0 km
32 km	69.4	63.3	57.5	47.1
34	66.1	60.2	54.7	44.8
36	63.0	57.4	52.1	42.6
38	60.1	54.7	49.6	40.5
40	57.4	52.2	47.3	38.5
42	54.9	49.8	45.1	36.7
44	52.6	47.7	43.1	35.0
46	50.4	45.7	41.3	33.5
48	48.4	43.8	39.6	32.1
50	46.5	42.1	38.0	30.7
52	44.8	40.5	36.5	29.5
54	43.2	39.0	35.2	28.4
56	41.7	37.6	33.9	27.3
58	40.3	36.3	32.7	26.3
60	38.9	35.1	31.6	25.4
62 km	37.7	34.0	30.6	24.5
64	36.5	32.9	29.6	23.7
66	35.4	31.9	28.7	23.0
68	34.4	31.0	27.8	22.2
70	33.4	30.1	27.0	21.6
72	32.5	29.2	26.2	21.0
74	31.6	28.4	25.5	20.4
76	30.8	27.7	24.8	19.8
78	30.0	27.0	24.2	19.3
80	29.2	26.3	23.6	18.8
82	28.5	25.6	23.0	18.3
84	27.9	25.0	22.4	17.8
86	27.2	24.4	21.9	17.4
88	26.6	23.9	21.4	17.0
90	26.0	23.3	20.9	16.6
$Z > 90$	$\frac{2340}{Z}$	$\frac{2076}{Z-1}$	$\frac{1837}{Z-2}$	$\frac{1426}{Z-4}$

These three summations are performed over the boundaries between sea level and flying height, between sea level and ground level, and between ground level and flying height, respectively.

Substituted in equation (4.8), this gives

$$\alpha = \frac{Z_c}{Z_c - Z_g}\,\alpha_c - \frac{Z_g}{Z_c - Z_g}\,\alpha_g - .000226 \tan\theta \, \frac{Z_g}{Z_c - Z_g}\,\Delta\rho \qquad (4.9)$$

where α_c and α_g are the photogrammetric refraction at the actual flying height and at the actual ground height, both with respect to a ground height at sea level, and $\Delta\rho$ is the difference between the densities at the actual flying height and ground height.

4.2 Contribution of earth curvature

A complication which has not been considered yet is the fact that, due to the earth curvature, the verticals in any two points of a tilted light ray are not parallel.

Since the angle θ is defined as the angle which the ray makes with the vertical, it follows that along a ray this angle varies not only because of the small atmospheric refraction but also because of the much larger change in the direction of the vertical.

As a result, $\tan\theta$ in equation (4.8) is not a constant, but is different for each term in the summation. If θ_c is the angle θ at the camera and β is the angle between the verticals at the camera and at an arbitrary point on the ray,

$$\theta = \theta_c - d\theta + \beta$$

In first approximation, and neglecting $d\theta$,

$$\tan\theta = \tan\theta_c + \sec^2\theta_c\,\beta$$

or, again approximately,

$$\tan\theta = \tan\theta_c \left(1+\sec^2\theta_c \, \frac{Z_c - Z}{R}\right) \qquad (4.10)$$

where R is the radius of the earth.

This formula can be derived also by differentiating the formula for the refraction in the atmosphere

$$n\,R\,\sin\theta = \text{constant}.$$

It follows from equation (4.10) that $\tan\theta$ in equation (4.8) should be replaced by the expression in equation (4.10), and the second factor in this expression should be brought under the summation. The same result is obtained by replacing $\tan\theta$ by $\tan\theta_c$ and adding

$$\Sigma\left[\sec^2\theta_c \frac{Z_c - Z}{R} (Z-Z_g)d\rho\right]$$

to the sum of products.

The computation of the sum for all flying heights and for all ground heights can be simplified by writing it as

$$\sec^2\theta_c \frac{Z_c - Z_g}{R} \Sigma((Z-Z_g)d\rho) - \Sigma\left[\sec^2\theta_c \frac{Z-Z_g}{R} (Z-Z_g)d\rho\right]$$

in which only the simple first term is a function of the flying height.

The resulting corrections to the photogrammetric refraction for an angle $\theta = 45°$ are listed in Table 3 as a function of flying height and ground height. The corrections for other angles can be obtained from the table values by multiplication by $\frac{1}{2}\sec^2\theta$.

Table 3. Contribution of earth curvature to refraction for a ray at 45° with the vertical, in microradians

Flying height above sea level	Contribution to the refraction for ground heights of 0.0 km	1.0 km	2.0 km	4.0 km
5.0 km	0.03	0.02	0.01	0.00
10.0	0.10	0.07	0.06	0.03
15.0	0.18	0.15	0.12	0.08
20.0	0.26	0.23	0.19	0.14
25.0	0.33	0.29	0.25	0.18
30.0	0.39	0.34	0.30	0.22

The photogrammetric refraction, including the effect of earth curvature, is therefore derived from the table value c_1 found in Table 1 and the table value c_2 found in Table 2 by means of the equation

$$\alpha = \tan\theta\,(c_1 + \tfrac{1}{2}\sec^2\theta\, c_2) \qquad (4.11)$$

The value c_2 is so small that its contribution can practically always be neglected.

4.3 Refraction in the actual atmosphere

The last problem that has to be dealt with is the difference between the actual atmosphere and the standard atmospheres.

The temperature, pressure, and composition of the actual atmosphere are never known completely. Even if they were, the computation of the photogrammetric refraction would be too complicated.

Therefore, usually the difference between actual atmosphere and standard atmosphere is neglected.

An estimate of the photogrammetric refraction based on measurements of temperature, pressure, and relative humidity at ground level can be computed as follows.

The standard atmospheres have a temperature of 15°C (59°F) and a pressure of 760 mm Hg at sea level and they are dry. According to the law of Boyle for a perfect gas, an increase in the absolute temperature of 1 3/4% throughout the atmosphere, that is 5°C (9°F) at sea level, causes a uniform decrease of about 1 3/4% in the density. Therefore, it causes the same decrease in the density differences in equation (4.8) and in the photogrammetric refraction. An increase of 1 1/3% in the pressure, that is 10 mm Hg at sea level, causes an increase of 1 1/3% in the density and in the photogrammetric refraction.

The density of damp air can be found by multiplying the density of dry air by the factor $(P-0.378p)/P$, where P is the pressure of the dry air and p is the pressure of the water vapour. With this formula, and a table of the saturation pressure of water vapour as a function of the temperature, the decrease in density can be computed as a function of the relative humidity. If the lower layers of the actual atmosphere have 100% relative humidity, while the pressure and temperature are the same as in the standard atmospheres, the decrease in density is 2/3% at sea level and 1/3% at 2000 m. Due to this variation in the decrease, the density differences and the photogrammetric refraction below 2000 m decrease by about 2%. If also the absolute temperatures are 3.5% higher than in the standard atmospheres, it is then 25°C (77°F) at sea level and the decrease in density is 4.5% at sea level and 4.0% at 2000 m. As a result, the density differences and the photogrammetric refraction decrease by about 7%.

5. Earth curvature

When the strip coordinates obtained by analytical strip triangulation are directly transformed to a three-dimensional rectangular coordinate system constructed from map coordinates (as easting and northing) and heights, one obstacle is met. This consists in the fact that the model of the earth presented by this coordinate system is deformed: on the earth the height of a point is the shortest distance from the point to the curved equipotential surface at sea level and in this coordinate system it is the shortest distance to the plane which contains the horizontal axes.

The deviation of the equipotential surface from a plane is considerable, even in the area of one strip. A 230 by 230 mm photograph taken with a 152.4 mm (6") lens at a height of 6 km covers an area of 9 by 9 km. Points on this surface in the middle of the sides of this area are already 1.6 m below the plane which is tangent to the surface in the centre. Points in the corners are 3.2 m below this plane. If the photograph is the first one of a strip that is 100 km long, points at the end of the strip are 800 m below the plane.

It follows that even for a single photograph this deviation cannot be neglected. The direct transformation produces correct map coordinates and heights only if it is preceded by or accompanied by a deformation of the triangulated strip that is identical with the deformation in that system.

In the area of a strip the equipotential surface may be approximated by the sphere which has as its radius the mean radius of the spheroid in this area. This sphere must be transformed into a plane. A satisfactory transformation is obtained by changing the great circle along the axis of the strip into a straight line of true length and by changing great circles at right angles to the first into straight lines of true length at right angles to the first. This can be done in two steps: first the sphere is changed into the cylinder which is tangent to the sphere in the great circle along the axis of the strip and then the cylinder is rolled out onto a plane which is tangent to the sphere in a point of that great circle.

This deformation can be obtained directly by giving appropriate corrections to the photograph coordinates. To derive these corrections for each photograph, the above plane is chosen tangent to the surface in the nadir point of the photograph.

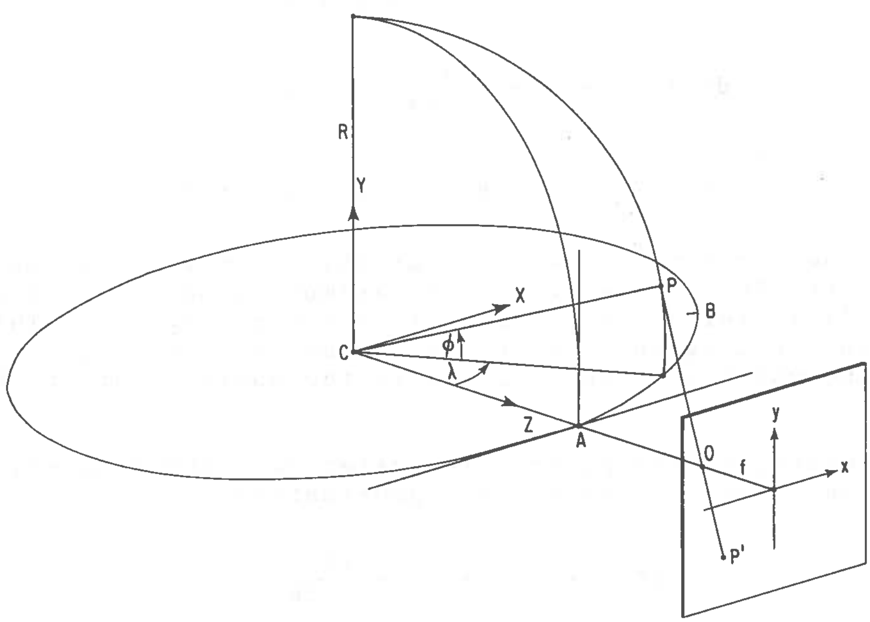

Figure 3. Elimination of the effect of earth curvature by projection of a sphere upon a tangent plane.

Figure 3 shows this situation. AB is the great circle along the axis of the strip and A is the nadir point of the photograph. A rectangular coordinate system is assumed with the origin in the centre of the sphere, the Z-axis through A, and the X-axis in the plane of the great circle.

If the great circle is the equator in a system of geographic coordinates ϕ and λ as shown in Figure 3 and the radius is called R, the geocentric coordinates of a point P on the sphere are:

$$X = R \cos\phi \sin\lambda$$
$$Y = R \sin\phi$$
$$Z = R \cos\phi \cos\lambda \qquad (5.1)$$

After the transformation of the sphere the point P is situated in the plane. Its coordinates are here

$$X_p = R\lambda$$
$$Y_p = R\phi$$
$$Z_p = R \quad , \qquad (5.2)$$

in which ϕ and λ are expressed in radians.

The corrections to the geocentric coordinates which transform the sphere into the plane are thus

$$dX = X_p - X = X \left(\frac{\lambda}{\cos\phi \sin\lambda} - 1\right)$$

$$dY = Y_p - Y = Y \left(\frac{\phi}{\sin\phi} - 1\right)$$

$$dZ = Z_p - Z = R\,(1 - \cos\phi \cos\lambda) \qquad (5.3)$$

For the above-mentioned photograph, the maximum value of ϕ and of λ is about 150". This makes the maximum values of dX and dY less than one millimeter. These corrections are negligible. This means that for each photograph the sphere may be projected orthogonally on the plane which is tangent to it in the nadir point of the photograph.

Therefore, in the geocentric system only the Z-coordinate needs a correction. With a negligible approximation,

$$dZ = \tfrac{1}{2} R(\phi^2+\lambda^2) = \frac{X^2 + Y^2}{2R} \qquad (5.4)$$

This correction has been derived for points on the sphere. However, because before the triangulation the heights are not known, points with different elevations will be given the same correction. As a result, a line which is perpendicular to the sphere will be shifted only. Consequently, it will not become orthogonal to the plane, as should be the case.

In the corners of the area covered by the above-mentioned photograph, the resulting errors are smaller than 0.1 m for every 100 m of difference in terrain height. They cause no y-parallaxes in a pair of photographs and, therefore, no errors in the relative orientation. They do cause x-parallaxes and, as a result, a small exaggeration of the vertical scale.

The errors could be corrected by taking heights, computed after the last-but-one iteration of the relative orientation, into account. This will be worthwhile if the corrections for asymmetric lens distortion are also taken into consideration.

6. Corrections for lens distortion, refraction, and earth curvature

Corrections to the photograph coordinates for symmetrical radial lens distortion are computed with the formulas

$$dx = x\,\frac{dr}{r} \text{ and } dy = y\,\frac{dr}{r} \tag{6.1}$$

where dr is the radial correction.

Corrections for refraction and earth curvature are computed with the same formulas after conversion of the photogrammetric refraction α and the earth curvature correction dZ to radial corrections.

The radial correction for refraction is

$$dr = -\frac{f}{\cos^2\theta}\,\alpha \tag{6.2}$$

Since $\alpha = c_1 \tan\theta$, it follows that

$$\frac{dr}{r} = -\left(1 + \frac{r^2}{f^2}\right) c_1 \tag{6.3}$$

Here, c_1 is the photogrammetric refraction in radians at $\theta = 45°$, taken from Table 2, and f is the focal length.

The radial correction for earth curvature can be computed with the help of Figure 4 which shows a vertical cross section through the sphere. It follows from similar triangles that

$$\left(\frac{H}{f}\,dr\right)/dZ = r/f. \tag{6.4}$$

Replacing dZ by the expression in the last part of equation (5.4), this gives

$$\frac{dr}{r} = \frac{H}{2R}\,\frac{r^2}{f^2} \tag{6.5}$$

The FORTRAN program derives the radial corrections for lens distortion from a table in which the value of dr is listed at a number of values of r. For each point, it computes dr by linear interpolation between the two nearest table values. The radial corrections for refraction are computed only if the value of c_1 is punched in the first card for a strip triangulation, and the radial corrections for earth curvature are computed only if the flying

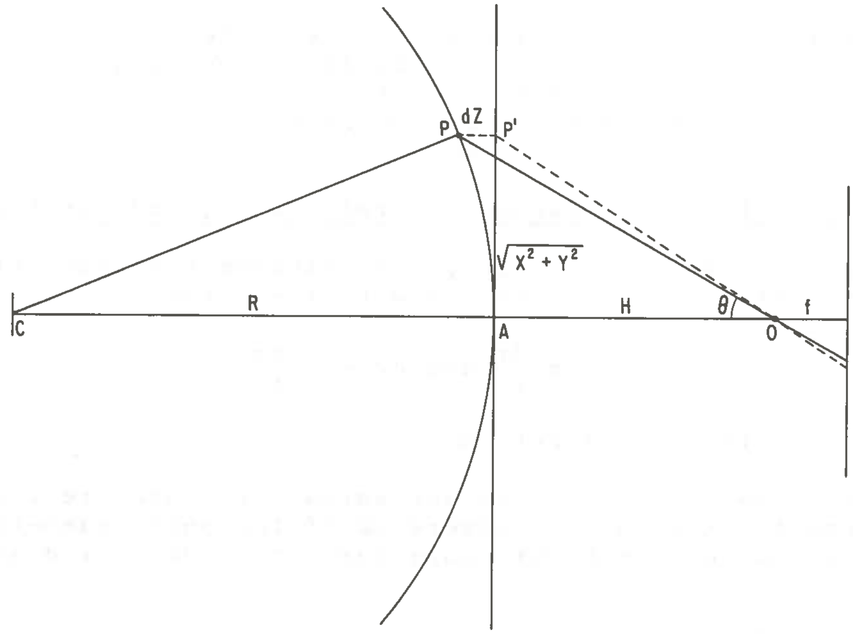

Figure 4. The radial correction for earth curvature.

height is punched in that card. Finally, the equations (6.1) are used to compute the corrections dx and dy from the sum of the dr/r.

The above corrections for refraction and earth curvature are valid for exactly vertical photographs. If the photographs are tilted, the corrections can be derived in the following way.

The angular correction α_1 for photogrammetric refraction is, according to equation (4.11), $\alpha_1 = c_1 \tan\theta$; the angular correction α_2 for earth curvature can be computed with the help of Figure 4 and equation (5.4) and is

$$\alpha_2 = dZ \sin\theta / \sqrt{H^2 + X^2 + Y^2} = \frac{H}{2R} \frac{\sin^3\theta}{\cos\theta} \qquad (6.6)$$

If the tilt of a photograph is τ, a ray in the direction of this tilt makes an angle $\theta - \tau$ with the camera axis. Therefore, its radial corrections for refraction and earth curvature can be computed with equation (6.2) if α is replaced by α_1 and by $-\alpha_2$, respectively, and θ is replaced by $\theta - \tau$.

As an example, the radial corrections in the direction of greatest tilt have been computed for the above-mentioned wide-angle photograph and for a super-wide angle photograph, assuming tilts of 0° and 2°. The resulting values are listed in Table 4. The change in the radial corrections is approximately proportional to the tilt.

The tilt of a "vertical" photograph is seldom more than 2°. If the corrections are based upon the assumption of exactly vertical

Table 4. Effect of tilt on radial corrections for refraction and for earth curvature

Angular distance from camera axis	Radial corrections for refraction		Radial corrections for earth curvature	
	tilt 0°	tilt 2°	tilt 0°	tilt 2°
	f = 152.4 mm, H = 6000 m			
9°	- 1.5 μ	- 1.8 μ	0.3 μ	0.5 μ
18°	- 3.2	- 3.6	2.5	3.4
27°	- 5.7	- 6.2	9.5	11.7
36°	- 9.9	-10.7	27.5	32.5
45°	-17.9	-19.2	71.7	82.3
	f = 88.2 mm, H = 6000 m			
45°	-10.4	-11.1	41.5	47.6
59°	-32.5	-35.2	191.	216.

photographs, the error in the correction for refraction is thus not greater than one micron in the corners of the wide-angle photograph and less everywhere else. Considering the uncertainty in the refraction itself, this error may be neglected.

If the tilt is 2°, the corresponding errors in the correction for earth curvature in the corners of the wide-angle photograph are up to 10 microns. However, if two consecutive photographs have the same tilt the effect on the model is small. This may be seen by rotating photographs and model until the camera axes are vertical. Errors in the model are now caused only by the difference between the height H used in formula (6.5) and the heights of the projection centres above terrain points in the rotated model.

This shows that the absolute tilt of the photographs is of relatively little importance. The difference in tilt of successive photographs however must be small: less than half a degree to make the error in the correction for earth curvature less than 2 microns. Only then can formula (6.5) be applied safely to tilted photographs.

Consequently, if the accuracy requirements make it necessary to apply corrections for asymmetric lens distortion, the earth curvature corrections should take into account not only the height differences, as stated before, but also the tilt of the photographs in the strip coordinate system.

In the case of the above super-wide-angle photograph, the maximum errors in the correction for earth curvature are 25 microns. Therefore, here it seems to be better not to apply a symmetrical radial correction for earth curvature to the photograph coordinates. Either an asymmetrical correction should be used or the strip coordinates should be corrected instead.

III. The orientation of a photograph

1. The orthogonal transformation matrix

For each photograph of a strip, a three-dimensional rectangular coordinate system x, y, z will be assumed with the origin in the projection centre of the photograph. The x- and y-axes will be parallel to the plane of the photograph and, therefore, the x- and y-coordinates of an image point will be identical with the photograph coordinates. The z-coordinate of an image point will be equal to +f (the calibrated focal length) if the photograph is in negative position (above the projection centre) and equal to -f if the photograph is in positive position.

The strip triangulation will be performed with respect to a three-dimensional rectangular coordinate system. In this system, the orientation of each photograph can be defined by means of the three coordinates of the projection centre and of parameters which determine its attitude.

Further, for each photograph a coordinate system X, Y, Z is needed with the origin in the projection centre of the photograph and with axes parallel to the axes of the strip coordinate system.

The relation between the coordinates X, Y, Z and the coordinates x, y, z of any point in a photograph is given by the matrix equation

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

or, simply,

$$\mathbf{X} = \mathbf{A}\,\mathbf{x} \qquad (1.1)$$

This is immediately evident when in Figure 5 the components of the position vector of a point with respect to the x, y, z coordinate system are projected upon the X-axis, the Y-axis, and the Z-axis. The elements of the first, second and third column of the matrix $\mathbf{A}$ are then seen to be the direction cosines of the x-, y-, and z-axes with respect to the X, Y, Z coordinate system.

Equation (1.1) will be used to define the attitude of a photograph. Therefore, either the nine direction cosines can be selected as the parameters which define its attitude or the direction cosines must be defined as functions of other suitable parameters.

The matrix $\mathbf{A}$ has the property that

$$\mathbf{A}^T \mathbf{A} = \mathbf{I} \qquad (1.2)$$

The superscript T indicates the transpose of the matrix to which it is attached, and the matrix $\mathbf{I}$ is the unit matrix.

This can be proved as follows. The position vectors $\mathbf{X}$ and

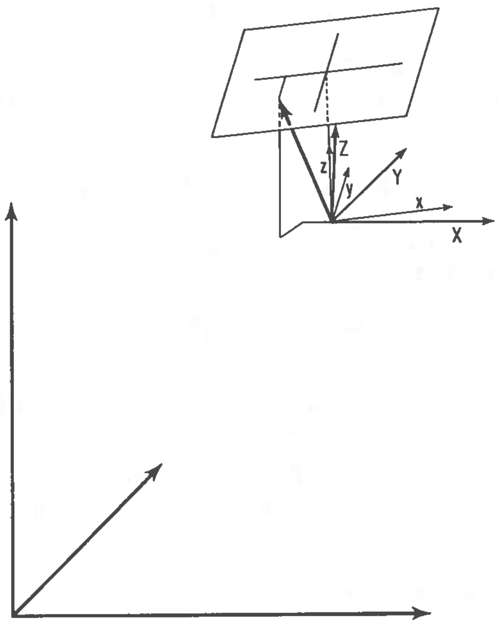

Figure 5. Coordinate systems used for the orientation of a photograph, and the position vector of an image point with its components.

$\mathbf{x}$ are identical, the only difference being the coordinate system in which their components are defined. Therefore, they have the same length, and consequently the sum of the products of their components is the same. In matrix notation:

$$\mathbf{X}^T \mathbf{X} = \mathbf{x}^T \mathbf{x} \quad ,$$

or

$$\mathbf{x}^T \mathbf{A}^T \mathbf{A}\, \mathbf{x} = \mathbf{x}^T \mathbf{x} \quad ,$$

and

$$\mathbf{x}^T (\mathbf{A}^T \mathbf{A} - \mathbf{I})\, \mathbf{x} = 0 \quad ,$$

for any vector $\mathbf{x}$. The first part of this equation is a polynomial of the second degree in the components of $\mathbf{x}$. It is equal to zero for any vector $\mathbf{x}$ if and only if all coefficients in this polynomial are equal to zero, that is, if $\mathbf{A}^T \mathbf{A} - \mathbf{I} = \mathbf{O}$!

Equation (1.2) states that the sum of the squares of the elements of each column of $\mathbf{A}$ is equal to 1 and that the sum of the products of corresponding elements in different columns is equal to zero. A matrix which satisfies these conditions is called orthogonal. Thus, the equation contains six independent relations between the nine elements of $\mathbf{A}$. Therefore, the elements contain only three independent parameters.

It follows from equation (1.2) that

$$\mathbf{A}^{-1} = \mathbf{A}^{T} \tag{1.3}$$

and therefore also that $\mathbf{A}\,\mathbf{A}^{T} = \mathbf{I}$. The latter equation expresses similar relations which exist between the elements of the rows of $\mathbf{A}$.

These relations can be expressed by the following theorems:

I. The sum of the squares of the elements in each column and in each row is equal to 1:

$$a_{1i}^2 + a_{2i}^2 + a_{3i}^2 = 1$$

$$a_{i1}^2 + a_{i2}^2 + a_{i3}^2 = 1 \qquad (i=1,2,3)$$

II. The sum of the products of the elements in corresponding positions in each two columns and in each two rows is equal to zero:

$$a_{1i}a_{1j} + a_{2i}a_{2j} + a_{3i}a_{3j} = 0$$

$$a_{i1}a_{j1} + a_{i2}a_{j2} + a_{i3}a_{j3} = 0 \qquad (i=1,2,3;\ j=1,2,3;\ i \neq j)$$

From these two theorems, two other theorems can be derived:

III. Each element is equal to its cofactor:

$$a_{il} = a_{jm}a_{kn} - a_{jn}a_{km}$$

Here, the row indices i, j, and k represent three sequential numbers of the sequence 1, 2, 3, 1, 2 and the column indices l, m, and n independently represent three sequential numbers of this sequence.

IV. The determinant of the matrix is equal to +1.

The last two theorems are valid only if the two coordinate systems are either both right-handed, as in the present case, or both left-handed. The matrix is then called proper orthogonal. If one of the coordinate systems is right-handed and the other is left-handed, each element is equal in magnitude to its cofactor but of opposite sign and the determinant is equal to -1.

The transformation (1.1) has here been introduced as representing the change in the components of a position vector under a change in the choice of coordinate system. However, by drawing a position vector which has components x, y, and z with respect to the X, Y, Z system, it becomes evident that the transformation also represents the rotation of the position vector from this position to the one in Figure 5. Therefore, it can represent the rotation of the photograph about the projection centre. The coordinate system X, Y, Z is then the only one and does not change its position.

Let now two such rotations be applied in succession:

$\mathbf{X}_1 = \mathbf{A}_1 \mathbf{x}$ and $\mathbf{X}_2 = \mathbf{A}_2 \mathbf{X}_1$. It follows from these equations that the final position vector of any image point can be found directly from $\mathbf{X}_2 = \mathbf{A}_2 \mathbf{A}_1 \mathbf{x}$. Since the two rotations do not change the length of any vector, the matrix $\mathbf{A} = \mathbf{A}_2 \mathbf{A}_1$ which represents the one-step rotation from $\mathbf{x}$ to $\mathbf{X}_2$ is also orthogonal. Consequently, the product of two orthogonal matrices is again an orthogonal matrix.

While thus to any attitude of the photograph corresponds one proper orthogonal matrix $\mathbf{A}$, it is also true that to every proper orthogonal matrix of the third order corresponds one attitude of the camera. For, since the sum of the squares of the elements in each column is equal to 1, these elements are the direction cosines of axes x, y, and z. Since the sum of products of corresponding elements in each two different columns is equal to zero, these axes are mutually orthogonal. Since the determinant of the matrix is equal to +1, this x, y, z system and the X, Y, Z system are either both right-handed or both left-handed. Therefore, this x, y, z system and the x, y, z system in equation (1.1) are identical.

Consequently, there is a one-to-one correspondence between the possible attitudes of a photograph and the proper orthogonal matrices of the third order. Any proper orthogonal matrix of the third order constructed in any way from three independent parameters can serve as the matrix for equation (1.1).

The constructions which are most important in analytical triangulation are described in the following. A much more complete account can be found in reference [19].

2. Rotations about three mutually orthogonal axes

The matrices

$$\mathbf{R}_\omega = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\omega & -\sin\omega \\ 0 & \sin\omega & \cos\omega \end{bmatrix}, \quad \mathbf{R}_\phi = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix}, \text{ and } \mathbf{R}_\kappa = \begin{bmatrix} \cos\kappa & -\sin\kappa & 0 \\ \sin\kappa & \cos\kappa & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (2.1)$$

satisfy theorems I and II and their determinants are equal to +1. Therefore, they are proper orthogonal.

Used as the matrix of equation (1.1), each leaves one coordinate unchanged and therefore can move any point only in a plane which is orthogonal to the axis of the unchanged coordinate. Also, it leaves the distance from the point to this axis unchanged.

Consequently, $\mathbf{R}_\omega$, $\mathbf{R}_\phi$, and $\mathbf{R}_\kappa$ represent rotations about the X-, Y-, and Z-axis, respectively. The signs of the elements are such that a positive rotation of three-dimensional space with respect to the coordinate system means a clockwise rotation when viewing in the positive direction of its axis.

Any matrix which is the product of three matrices $\mathbf{R}_\omega$, $\mathbf{R}_\phi$, and $\mathbf{R}_\kappa$ is, according to the preceding, also orthogonal and it contains the required number of three evidently independent parameters.

If first a rotation κ, then a rotation ϕ, and finally a rotation ω is applied, the resulting attitude of the photograph is represented by the matrix

$$\mathbf{A} = \mathbf{R}_{\omega}\ \mathbf{R}_{\phi}\ \mathbf{R}_{\kappa} \tag{2.2}$$

Matrix multiplication gives

$$\mathbf{A} = \begin{bmatrix} \cos\phi\cos\kappa & -\cos\phi\sin\kappa & \sin\phi \\ a_{21} & a_{22} & -\sin\omega\cos\phi \\ a_{31} & a_{32} & \cos\omega\cos\phi \end{bmatrix} \tag{2.3}$$

with

$$a_{21} = \sin\omega\sin\phi\cos\kappa + \cos\omega\sin\kappa$$

$$a_{22} = -\sin\omega\sin\phi\sin\kappa + \cos\omega\cos\kappa$$

$$a_{31} = -\cos\omega\sin\phi\cos\kappa + \sin\omega\sin\kappa$$

$$a_{32} = \cos\omega\sin\phi\sin\kappa + \sin\omega\cos\kappa$$

The cameras of photogrammetric plotting instruments have mutually orthogonal axes. If the κ-axis is the tertiary axis, the κ-rotation is applied about a vertical Z-axis only if at that time the rotations ω and ϕ are equal to zero. If the ϕ-axis is the secondary axis and the ω-axis is the primary axis, a ϕ-rotation is applied about a horizontal axis only if at that time the ω-rotation is equal to zero. Therefore, the matrix in equation (2.2) represents the orientation matrix for an instrument with a tertiary κ-axis, a secondary ϕ-axis and a primary ω-axis. This matrix has been recommended for use in analytical photogrammetry in a resolution of the International Society for Photogrammetry, adopted at the 1960 Congress.

Each of the six possible arrangements of successive transformations $\mathbf{R}_{\omega}$, $\mathbf{R}_{\phi}$, and $\mathbf{R}_{\kappa}$ corresponds to one of the six possible choices of primary, secondary, and tertiary axes. Each leads to an orientation matrix in which one of the off-diagonal elements is equal to the sine of the secondary rotation.

Obviously, orthogonal matrices are obtained also if in any one of the matrices (2.1) the minus sign is attached to the other sine. This changes the positive direction of a rotation.

By changing the arrangement of the matrices and the position of the minus signs, 48 different proper orthogonal matrices can be constructed from three rotations. Each corresponds to a certain choice of primary, secondary and tertiary axis and of positive directions of the rotations.

Because of the one-to-one correspondence between the possible attitudes of a photograph and the proper orthogonal matrices of order three, for a given attitude of the photograph these 48 matrices are numerically the same. The difference consists in the formulation of the elements as functions of three parameters and in the values of these parameters.

In the case of analytical aerial triangulation, where the camera axis at the moment of exposure is usually nearly vertical and the x-axis of each photograph can be chosen roughly parallel to the X-axis of the strip coordinate system, the three rotations in each of these matrices are only small. This makes these matrices suitable for use in the iterative procedure of relative orientation. From the mathematical point of view, they are all equally useful.

If in each of these matrices a minus sign is attached to the cosine of an angle, if sine and cosine of an angle are interchanged, or if more than one of these changes are introduced simultaneously, proper orthogonal matrices are obtained also. Geometrically, each of these changes means replacing a small angle by an angle which is closer to 90°, 180°, or 270°. Consequently, these matrices are less useful for analytical aerial triangulation.

Orthogonal matrices are obtained also if the algebraic construction starts with an on-diagonal element which is equated to the cosine or the sine of an angle. Geometrically, this corresponds to applying the first and third rotations about the same axis, and applying the second rotation about one of the other two axes. If the first and third rotations are applied about the Z-axis, these rotations are known as swing of the photograph and azimuth of the principal plane, respectively. These matrices are not useful here because, if the second rotation is equal to zero, the first and third rotations are not defined. Also, when the second rotation is not equal to zero, the first and third rotations may have any value from 0° to 360°.

3. Rotation about a directed line

Starting from the position where the x, y, z axes coincide with the X, Y, Z axes, the correct attitude of a photograph can be established by means of a single rotation about a suitable axis through the origin of the two coordinate systems.

This statement is obviously true if the orientation leaves at least one point, besides the origin, in its initial position. The axis of rotation is then the line through this point and the origin.

In other words, the statement is true if always a set of values of x, y, and z can be found, not all three equal to zero, for which $\mathbf{A}\,\mathbf{x} = \mathbf{x}$. According to a theorem of linear algebra, this is the case if and only if $|\mathbf{A} - \mathbf{I}| = 0$. It can be proved that this determinant is indeed equal to zero by writing it as a polynomial in the nine elements of $\mathbf{A}$ and simplifying this expression by means of first theorem III, then theorem I.

Let the matrix $\mathbf{A}$ have the direction cosines λ, μ, and ν of the axis of rotation and the angle of rotation α as parameters. Since

$$\lambda^2 + \mu^2 + \nu^2 = 1,$$

the matrix will again contain three independent parameters. In terms of these parameters, the matrix can be derived by means of a theorem from matrix algebra which states that the rotation about a directed line can be represented by the matrix

$$\mathbf{A} = \mathbf{T}\ \mathbf{R}_{\omega}\ \mathbf{T}^{-1} \tag{3.1}$$

in which $\mathbf{T}$ is any proper orthogonal matrix with the direction cosines of the axis of rotation as the elements of the first column. Multiplication of the three matrices gives

$$\mathbf{A} = \begin{bmatrix} \lambda^2(1-\cos\alpha) + \cos\alpha & \lambda\mu(1-\cos\alpha) - \nu\sin\alpha & \lambda\nu(1-\cos\alpha) + \mu\sin\alpha \\ \lambda\mu(1-\cos\alpha) + \nu\sin\alpha & \mu^2(1-\cos\alpha) + \cos\alpha & \mu\nu(1-\cos\alpha) - \lambda\sin\alpha \\ \lambda\nu(1-\cos\alpha) - \mu\sin\alpha & \mu\nu(1-\cos\alpha) + \lambda\sin\alpha & \nu^2(1-\cos\alpha) + \cos\alpha \end{bmatrix} \tag{3.2}$$

where $\lambda^2 + \mu^2 + \nu^2 = 1$.
This matrix can be found in a number of mathematical textbooks.

This form of the orientation matrix can be derived also by means of vector theory.

Let, as shown in Figure 6, $\overline{OQ}$ be the position vector $\underline{x}$ of a point before a rotation α about the axis OP and let $\overline{OR}$ be its position vector $\mathbf{X}$ after the rotation. The vector $\mathbf{X}$ can be regarded as the sum of a vector $\overline{OT}$ along the vector $\underline{x}$, a vector $\overline{TS}$ parallel to the axis of rotation, and a vector $\overline{SR}$ perpendicular to $\underline{x}$ and to the axis of rotation.

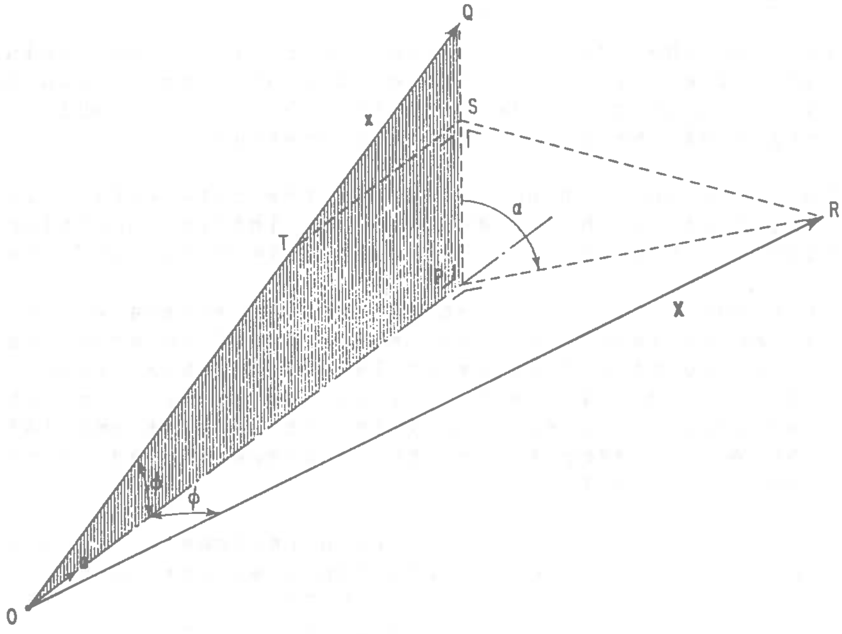

Figure 6. Rotation of a vector about a directed line.

Since the line element PS = PR cos α = PQ cos α, it follows that the line element OT = OQ cos α, and so

$$\overline{OT} = \mathbf{x} \cos \alpha . \qquad (3.3)$$

The dot product $\mathbf{a} \cdot \mathbf{x}$ of two vectors $\mathbf{a}$ and $\mathbf{x}$ is a scalar and is equal to the product of the lengths of the two vectors by the cosine of the angle between them. Therefore, if $\mathbf{a}$ is a unit vector along the axis OP, the length of the vector $\overline{OP}$ is $\mathbf{a} \cdot \mathbf{x}$ and so $\overline{OP} = (\mathbf{a} \cdot \mathbf{x})\mathbf{a}$ and

$$\overline{TS} = \overline{OP}\,(1 - \cos \alpha) = (\mathbf{a} \cdot \mathbf{x})\,\mathbf{a}\,(1 - \cos \alpha) \qquad (3.4)$$

The cross product $\mathbf{a} \times \mathbf{x}$ of the two vectors $\mathbf{a}$ and $\mathbf{x}$ is a vector parallel to and with the same positive direction as the vector $\overline{SR}$. Its length is the product of the lengths of the two vectors and the sine of the angle between them. Since the line element SR = PR sin α = PQ sin α = OQ sin ϕ sin α,

$$\overline{SR} = \mathbf{a} \times \mathbf{x} \sin \alpha \qquad (3.5)$$

Summation of equations (3.3), (3.4) and (3.5) gives

$$\mathbf{X} = \mathbf{x} \cos \alpha + \mathbf{a}\,(\mathbf{a} \cdot \mathbf{x})(1 - \cos \alpha) + \mathbf{a} \times \mathbf{x} \sin \alpha \qquad (3.6)$$

In this equation, the components of the vector $\mathbf{x}$ are the coordinates x, y, and z. The components of the unit vector $\mathbf{a}$ are the direction cosines λ, μ, and ν of the axis of rotation. Further, the dot product $\mathbf{a} \cdot \mathbf{x}$ is the sum of the products of the corresponding components:

$$\mathbf{a} \cdot \mathbf{x} = \lambda x + \mu y + \nu z \qquad (3.7)$$

The cross product $\mathbf{a} \times \mathbf{x}$ is a vector whose components are the cofactors of the elements of the first row of a matrix which has the components of $\mathbf{a}$ and of $\mathbf{x}$ as the elements of its second and third row, respectively:

$$\mathbf{a} \times \mathbf{x} = \begin{bmatrix} \mu z - \nu y \\ \nu x - \lambda z \\ \lambda y - \mu x \end{bmatrix} \qquad (3.8)$$

By substituting these expressions for the two vector products in equation (3.6), that equation can be written in terms of vector components. This shows that the equation is equivalent to equation (1.1), if $\mathbf{A}$ in that equation is the matrix of equation (3.2).

4. A purely algebraic derivation

The preceding derivations of orthogonal matrices were based upon the concept of the orientation of a photograph by means of one or more rotations. Although in the case of a matrix constructed from three rotations a construction was developed in which this concept was not directly employed, the interpretation of the parameters as being three rotations was always possible.

In analytical photogrammetry, however, there is no need for such an interpretation. It has the disadvantages of necessitating the computation of trigonometric functions. The following method allows the construction of the elements of the orthogonal matrix as rational functions of three independent parameters.

The skew-symmetric matrix with real elements

$$\mathbf{S} = \begin{bmatrix} o & -c & b \\ c & o & -a \\ -b & a & o \end{bmatrix} \qquad (4.1)$$

contains three independent parameters, as does the proper orthogonal matrix **A** of order three. Therefore, to each **S** corresponds one proper orthogonal matrix **A** with the same parameters.

A matrix **A** can be formulated in several ways as an analytical function of **S** . Choosing a fourth parameter d, one can write

$$\mathbf{A} = (d\,\mathbf{I} + \mathbf{S})(d\,\mathbf{I} - \mathbf{S})^{-1} \qquad (4.2)$$

The matrix $(d\,\mathbf{I} - \mathbf{S})^{-1}$ can be found by computing the matrix of co-factors of $d\,\mathbf{I} - \mathbf{S}$, transposing this matrix, and dividing its elements by the determinant of $d\,\mathbf{I} - \mathbf{S}$. Alternatively, the elements of the inverse can be computed from the 3 × 3 equations contained in $(d\,\mathbf{I} - \mathbf{S})(d\,\mathbf{I} - \mathbf{S})^{-1} = \mathbf{I}$. Following this, multiplication of the two matrices in the right-hand side of equation (4.2) gives:

$$\mathbf{A} = \begin{bmatrix} d^2+a^2-b^2-c^2 & 2ab-2cd & 2ac+2bd \\ 2ab+2cd & d^2-a^2+b^2-c^2 & 2bc-2ad \\ 2ac-2bd & 2bc+2ad & d^2-a^2-b^2+c^2 \end{bmatrix} \frac{1}{d^2+a^2+b^2+c^2} \qquad (4.3)$$

This matrix satisfies the previously mentioned theorems I, II, III, and IV and is, therefore, proper orthogonal.

A different formulation of **A** as a function of **S** can be obtained by writing $\mathbf{A} = (\mathbf{A}^T)^{-1}$. Substituting the expression (4.2) into the second part of this equation gives:

$$\mathbf{A} = (d\,\mathbf{I} - \mathbf{S})^{-1}(d\,\mathbf{I} + \mathbf{S}) \qquad (4.4)$$

Since this equation is equivalent to (4.2), it also leads to the equation (4.3).

Multiplication of all four parameters by the same factor does not alter the value of the elements of **A** . Therefore, **A** contains only three independent parameters.

Accordingly, it is possible to multiply the parameters by a factor which makes $d^2+a^2+b^2+c^2 = 1$ and makes d positive if it is negative. This reduces the matrix to the simpler but not less general form

$$\mathbf{A} = \begin{bmatrix} d^2+a^2-b^2-c^2 & 2ab-2cd & 2ac+2bd \\ 2ab+2cd & d^2-a^2+b^2-c^2 & 2bc-2ad \\ 2ac-2bd & 2bc+2ad & d^2-a^2-b^2+c^2 \end{bmatrix} \qquad (4.5)$$

in which $d^2+a^2+b^2+c^2 = 1$ and $d > 0$.

It is also possible to divide all four parameters by d. This simplifies the matrix to

$$\mathbf{A} = \begin{bmatrix} 1+a^2-b^2-c^2 & 2ab-2c & 2ac+2b \\ 2ab+2c & 1-a^2+b^2-c^2 & 2bc-2a \\ 2ac-2b & 2bc+2a & 1-a^2-b^2+c^2 \end{bmatrix} \frac{1}{1+a^2+b^2+c^2} \qquad (4.6)$$

This form of the orthogonal matrix can be obtained directly from

$$\mathbf{A} = (\mathbf{I} + \mathbf{S})(\mathbf{I} - \mathbf{S})^{-1}.$$

This formulation can be found in textbooks on linear algebra, usually with the signs interchanged.

This form appears to be the simplest one for electronic computation and is the one which is used in the FORTRAN program.

IV. Relative Orientation

1. The elements of relative orientation

Let the strip triangulation be performed with respect to a three-dimensional rectangular coordinate system X, Y, Z as shown in Figure 7. For each photograph, auxiliary coordinate systems X_i, Y_i,

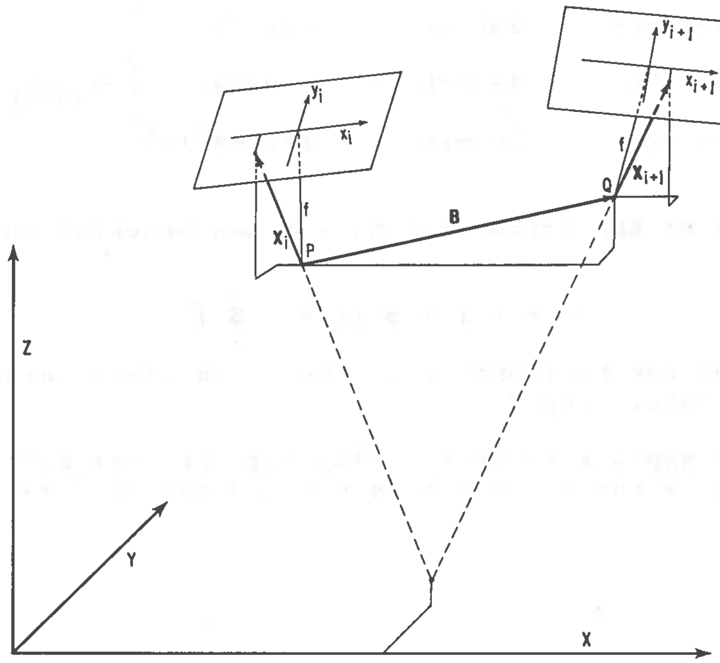

Figure 7. Vectors and vector components used in relative orientation.

Z_i and x_i, y_i, z_i which have their origin in the projection centre will be required. These are the two systems used in the preceding chapter, with a subscript added to indicate the number of the photograph. Therefore, the X_i-, Y_i-, and Z_i-axes are parallel to the X-, Y-, and Z-axes and the x_i- and y_i-axes are parallel to the plane of the photograph.

The projection centre of the first photograph will be given arbitrary coordinates in the X, Y, Z system and the coordinate axes x_1, y_1, and z_1 of this photograph will be made to coincide with the X_1-, the Y_1-, and the Z_1-axis, respectively. This completes the orientation of the first photograph.

The strip triangulation will be performed by computing in succession the relative orientation of each following photograph with respect to the preceding one. Each relative orientation will be followed by scaling of the resulting model and by computation of the strip coordinates X, Y, and Z of the measured points.

To determine the relative orientation of each photograph with respect to the preceding one, an arbitrary value is assumed for the base component b_X while the x-, y-, and z-axes of the photograph are first placed parallel to the X-, Y-, and Z-axes. The elements of relative orientation are then the base components b_Y and b_Z and three independent parameters which determine the orientation matrix of the photograph.

2. The condition equation for relative orientation

The relative orientation of a photograph (i+1) with respect to the preceding one (i) consists in positioning the photograph in such a way that rays from corresponding images in the two photographs intersect.

Analytically, this means that a condition equation which states that corresponding rays intersect must be satisfied. The condition equation can state this requirement in different ways. For instance, it can state that the two rays must be co-planar, that the minimum distance between the rays must be equal to zero, or, assuming that the strip axis is approximately parallel to the x-axis, that the Y-parallaxes must be equal to zero.

The requirement of co-planarity can be formulated as the condition that two corresponding image points and the two projection centres must lie in one plane. According to an equation from analytical geometry, in this case a fourth-order determinant which has the strip coordinates of these points as the elements of the first three columns is equal to zero:

$$\begin{vmatrix} X^P & Y^P & Z^P & 1 \\ X^Q & Y^Q & Z^Q & 1 \\ X^P + X_i & Y^P + Y_i & Z^P + Z_i & 1 \\ X^Q + X_{i+1} & Y^Q + Y_{i+1} & Z^Q + Z_{i+1} & 1 \end{vmatrix} = 0 \qquad (2.1)$$

In this equation, the strip coordinates of the projection centres of photographs i and i+1 are indicated by superscripts P and Q.

From this equation it follows by subtraction of rows that

$$\begin{vmatrix} b_X & b_Y & b_Z \\ X_i & Y_i & Z_i \\ X_{i+1} & Y_{i+1} & Z_{i+1} \end{vmatrix} = 0 \qquad (2.2)$$

This is the condition equation for relative orientation. The base components b_X, b_Y, and b_Z in the first row of the determinant are the differences between the strip coordinates of the two projection centres. The subscripted coordinates in the second and third rows are the components of the vectors $\mathbf{X}_i$ and $\mathbf{X}_{i+1}$ from projection centre to image point in photographs i and i+1. They are functions of the orientation matrices of the two photographs:

$$\mathbf{X}_i = \mathbf{A}_i \, \mathbf{x}_i$$

$$\mathbf{X}_{i+1} = \mathbf{A}_{i+1} \, \mathbf{x}_{i+1} \qquad (2.3)$$

Alternatively, the condition of co-planarity can state that the vectors $\mathbf{X}_i$ and $\mathbf{X}_{i+1}$ and the vector $\mathbf{B}$ from projection centre P to projection centre Q must lie in one plane. According to an equation from vector analysis, this means that their scalar triple product must be equal to zero:

$$\mathbf{B} \cdot \mathbf{X}_i \times \mathbf{X}_{i+1} = 0 \qquad (2.4)$$

This is the condition equation for relative orientation in vector notation.

If $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are unit vectors along X-, Y-, and Z-axis respectively, each of the three vectors in (2.4) can be written as the sum of vectors along these axes:

$$\mathbf{B} = b_X \mathbf{i} + b_Y \mathbf{j} + b_Z \mathbf{k} ,$$

$$\mathbf{X}_i = X_i \mathbf{i} + Y_i \mathbf{j} + Z_i \mathbf{k} ,$$

$$\mathbf{X}_{i+1} = X_{i+1} \mathbf{i} + Y_{i+1} \mathbf{j} + Z_{i+1} \mathbf{k} . \qquad (2.5)$$

With the help of equation (3.8) of the preceding chapter, the cross product $\mathbf{X}_i \times \mathbf{X}_{i+1}$ can now be written as a vector:

$$\mathbf{D} = (Y_i Z_{i+1} - Z_i Y_{i+1}) \mathbf{i} + (Z_i X_{i+1} - X_i Z_{i+1}) \mathbf{j} + (X_i Y_{i+1} - Y_i X_{i+1}) \mathbf{k} \qquad (2.6)$$

With the help of equation (3.7) of that chapter, the dot product $\mathbf{B} \cdot \mathbf{D}$ can be written as a scalar. According to equation (2.4), this scalar is equal to zero:

$$b_X(Y_i Z_{i+1} - Z_i Y_{i+1}) + b_Y(Z_i X_{i+1} - X_i Z_{i+1}) + b_Z(X_i Y_{i+1} - Y_i X_{i+1}) = 0 \qquad (2.7)$$

Since this equation is obtained also if in (2.2) the determinant is expanded in terms of the elements of the first row, the equations (2.2) and (2.4) are equivalent. In the following sections, equation (2.4) will be used rather than equation (2.2) because it allows a more compact presentation of the formulas.

3. Differentiation of the condition equation

For each pair of corresponding image points, a condition equation (2.4) can be formulated. To compute the elements of relative orientation, at least five such equations obtained from five or more pairs of points must be available.

The equations are not linear with respect to the five elements. This makes it impossible to solve them directly on digital electronic computers. Most of these computers can perform no other mathematical operations than addition, subtraction, multiplication, and division.

Since linear equations can be solved by means of those operations, the condition equations must be replaced by linear approximations. Those can be derived by differentiating the condition equations with respect to the five orientation elements. The differentiation produces equations which are linear with respect to corrections to assumed approximations of the orientation elements.

Since the linear equations are only approximations of the condition equations, their solution gives only approximate values of the required corrections. Adding these corrections to the assumed approximations of the orientation elements gives improved approximations. Those must then be substituted into the linear equations and new corrections must be computed. The procedure must be repeated until the corrections become negligible. Thus, the orientation elements are computed in an iterative procedure.

Before equation (2.4) is differentiated, a modification can be introduced which results in simpler coefficients in the linear equation. This modification consists in premultiplying the matrix $\mathbf{A}_{i+1}$ by an orthogonal matrix $\mathbf{R}$. This changes the equation to

$$\mathbf{B} \cdot \mathbf{X}_i \times (\mathbf{R}\,\mathbf{A}_{i+1}\,\mathbf{x}_{i+1}) = 0 \tag{3.1}$$

The matrix $\mathbf{R}$ will be constructed from three parameters in the same way as the matrix $\mathbf{A}_{i+1}$ is constructed from its parameters. The matrix $\mathbf{A}_{i+1}$ will be the matrix of the assumed approximate orientation and the matrix $\mathbf{R}$ will serve to correct it. Thus, the parameters of $\mathbf{R}$ will be used as unknowns in the linear equations instead of corrections to the parameters of $\mathbf{A}_{i+1}$.

This equation must now be differentiated with respect to the components b_Y and b_Z of $\mathbf{B}$ and the three parameters of $\mathbf{R}$. The differentiation requires the use of approximate values for these five variables. Those for the two base components are equal to the assumed approximations and those for the three parameters are equal to zero.

Differentiation of $\mathbf{R}$ gives

$$\mathbf{R} \approx \mathbf{I} + \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \tag{3.2}$$

Here, a_1, a_2, and a_3 are functions of the parameters of the orthogonal matrix. For each of the suitable forms of the orthogonal matrix described in the preceding chapter, these functions are listed in Table 5.

If now $d\mathbf{B}$ is the vector whose components are the required corrections 0, db_Y, and db_Z to the base components and $\mathbf{R}_1$ is the skew-symmetric matrix in equation (3.2), the differentiation of equation (3.1) has as its result equation (3.4).

This equation can be obtained without using the rules for differentiation of a scalar triple product by linearization of equation (3.1). First, the introduction of the correction vector

Table 5. Functions of the matrix parameters, to be solved from the condition equation (3.1)

Matrix in the preceding chapter	Parameters a_1	a_2	a_3
(2.3), derived from three rotations	ω	ϕ	κ
(3.2), derived from one rotation	$\lambda\alpha$	$\mu\alpha$	$\nu\alpha$
(4.5) and (4.6), derived from a skew-symmetric matrix	2a	2b	2c

$\mathbf{dB}$ and the linear approximation (3.2) of $\mathbf{R}$ gives:

$$(\mathbf{B} + \mathbf{dB}) \cdot \mathbf{X}_i \times (\mathbf{X}_{i+1} + \mathbf{R}_1 \mathbf{X}_{i+1}) = 0 \qquad (3.3)$$

This equation still contains products of the corrections to the base components and the parameters of $\mathbf{R}$. It is linearized by writing the scalar triple product as a sum of such products and omitting the term which contains both $\mathbf{dB}$ and $\mathbf{R}_1$. This gives

$$\mathbf{B} \cdot \mathbf{X}_i \times \mathbf{X}_{i+1} + \mathbf{dB} \cdot \mathbf{X}_i \times \mathbf{X}_{i+1} + \mathbf{B} \cdot \mathbf{X}_i \times (\mathbf{R}_1 \mathbf{X}_{i+1}) = 0 \qquad (3.4)$$

The product $\mathbf{R}_1 \mathbf{X}_{i+1}$ in this equation is a vector whose components can be obtained by performing the matrix multiplication. If $\mathbf{r}$ is the vector whose components are a_1, a_2, and a_3,

$$\mathbf{r} = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k} \,, \qquad (3.5)$$

the vectors $\mathbf{R}_1 \mathbf{X}_{i+1}$ and $\mathbf{r} \times \mathbf{X}_{i+1}$ are identical because they have the same components:

$$\mathbf{R}_1 \mathbf{X}_{i+1} = \begin{bmatrix} a_2 Z_{i+1} - a_3 Y_{i+1} \\ a_3 X_{i+1} - a_1 Z_{i+1} \\ a_1 Y_{i+1} - a_2 X_{i+1} \end{bmatrix} = \mathbf{r} \times \mathbf{X}_{i+1} \qquad (3.6)$$

Applied to equation (3.4), this gives

$$\mathbf{B} \cdot \mathbf{X}_i \times \mathbf{X}_{i+1} + \mathbf{dB} \cdot \mathbf{X}_i \times \mathbf{X}_{i+1} + \mathbf{B} \cdot \mathbf{X}_i \times (\mathbf{r} \times \mathbf{X}_{i+1}) = 0 \qquad (3.7)$$

The value of a scalar triple product does not change if the dot and the cross are interchanged or if the factors of the dot product are interchanged. If the factors of the cross product are interchanged, the sign changes. This is immediately evident when the corresponding operations are performed upon the rows of the determinant with which the scalar triple product is identical.

By means of such changes and a rearrangement of the terms, the equation can be brought in its final form:

$$\mathbf{X}_{i+1} \times (\mathbf{B} \times \mathbf{X}_i) \cdot \mathbf{r} + \mathbf{X}_i \times \mathbf{X}_{i+1} \cdot d\mathbf{B} + \mathbf{X}_i \times \mathbf{X}_{i+1} \cdot \mathbf{B} = 0 \qquad (3.8)$$

This equation is obviously linear with respect to the five unknowns: the components of $\mathbf{r}$ and of $d\mathbf{B}$. Their coefficients are the components of the two vectors $\mathbf{X}_{i+1} \times (\mathbf{B} \times \mathbf{X}_i)$ and $\mathbf{X}_i \times \mathbf{X}_{i+1}$. These can be computed as such, and therefore the equation is in a suitable form for use in electronic computation. It is used in this form in the FORTRAN program. The third term is the constant part; it is equal to zero if the vectors from the projection centres to the image points intersect.

The coefficients of the components of $\mathbf{r}$ can be written as scalar products as follows:

$$\begin{aligned} \mathbf{i} \cdot \mathbf{X}_{i+1} \times (\mathbf{B} \times \mathbf{X}_i) &= \mathbf{B} \times \mathbf{X}_i \cdot \mathbf{i} \times \mathbf{X}_{i+1} \\ \mathbf{j} \cdot \mathbf{X}_{i+1} \times (\mathbf{B} \times \mathbf{X}_i) &= \mathbf{B} \times \mathbf{X}_i \cdot \mathbf{j} \times \mathbf{X}_{i+1} \\ \mathbf{k} \cdot \mathbf{X}_{i+1} \times (\mathbf{B} \times \mathbf{X}_i) &= \mathbf{B} \times \mathbf{X}_i \cdot \mathbf{k} \times \mathbf{X}_{i+1} \end{aligned} \qquad (3.9)$$

One has:

$$\mathbf{i} \times \mathbf{X}_{i+1} = \begin{bmatrix} 0 \\ -Z_{i+1} \\ Y_{i+1} \end{bmatrix} \qquad \mathbf{j} \times \mathbf{X}_{i+1} = \begin{bmatrix} Z_{i+1} \\ 0 \\ -X_{i+1} \end{bmatrix} \qquad \mathbf{k} \times \mathbf{X}_{i+1} = \begin{bmatrix} -Y_{i+1} \\ X_{i+1} \\ 0 \end{bmatrix} \qquad (3.10)$$

Substitution of (3.10) into (3.9) gives the three coefficients as scalar triple products. These in turn can be written as determinants. In a similar way, the coefficients of the base components can be written as determinants. In this way, equation (3.8) is obtained in the form in which it appears in earlier publications of NRC:

$$\begin{vmatrix} b_X & b_Y & b_Z \\ X_i & Y_i & Z_i \\ 0 & -Z_{i+1} & Y_{i+1} \end{vmatrix} a_1 + \begin{vmatrix} b_X & b_Y & b_Z \\ X_i & Y_i & Z_i \\ Z_{i+1} & 0 & -X_{i+1} \end{vmatrix} a_2 + \begin{vmatrix} b_X & b_Y & b_Z \\ X_i & Y_i & Z_i \\ -Y_{i+1} & X_{i+1} & 0 \end{vmatrix} a_3 +$$

$$+ \begin{vmatrix} Z_i & X_i \\ Z_{i+1} & X_{i+1} \end{vmatrix} (b_Y + db_Y) + \begin{vmatrix} X_i & Y_i \\ X_{i+1} & Y_{i+1} \end{vmatrix} (b_Z + db_Z) + \begin{vmatrix} Y_i & Z_i \\ Y_{i+1} & Z_{i+1} \end{vmatrix} b_X = 0 \qquad (3.11)$$

4. Differentiation with respect to the photograph coordinates

If five points have been measured for relative orientation, the unknowns can be solved from the resulting five linear equations (3.8).

In practice, both as a check on errors and to improve the accuracy of the relative orientation, more than five points will be measured. This makes an adjustment necessary. For this, the method of least squares provides a convenient algorithm. In order not to complicate the computer program, this algorithm can be used even if only five equations are available.

The method of least squares requires that each equation (3.8) be given the proper weight. Determination of the weight requires differentiation of the condition equation (2.4) not only with respect to the five unknowns but also with respect to the measured quantities, which are here the photograph coordinates.

The differentiation can be performed by first adding corrections dx_i, dy_i, dx_{i+1}, and dy_{i+1} to the photograph coordinates. Thus, vectors

$$\mathbf{dx}_i = dx_i\,\mathbf{i} + dy_i\,\mathbf{j}$$

and

$$\mathbf{dx}_{i+1} = dx_{i+1}\,\mathbf{i} + dy_{i+1}\,\mathbf{j} \qquad (4.1)$$

are added to the vectors $\mathbf{x}_i$ and $\mathbf{x}_{i+1}$, respectively, in equations (2.3). Subsequently, the obtained expressions for $\mathbf{X}_i$ and $\mathbf{X}_{i+1}$ are substituted into equation (3.3) and this equation is linearized in the same way as before.

This differentiation results in two additional terms in the correction equation (3.8):

$$\mathbf{B}\cdot(\mathbf{A}_i\,\mathbf{dx}_i)\times\mathbf{X}_{i+1} \quad \text{and} \quad \mathbf{B}\cdot\mathbf{X}_i\times(\mathbf{A}_{i+1}\,\mathbf{dx}_{i+1})$$

When now in these terms $\mathbf{dx}_i$ and $\mathbf{dx}_{i+1}$ are replaced by the expressions in the equations (4.1), the two terms can be changed into four terms, each of which contains one of the four corrections to the photograph coordinates. When simultaneously the terms are brought to the second part of the correction equation, this equation becomes:

$$\mathbf{X}_{i+1}\times(\mathbf{B}\times\mathbf{X}_i)\cdot\mathbf{r} + \mathbf{X}_i\times\mathbf{X}_{i+1}\cdot\mathbf{dB} + \mathbf{X}_i\times\mathbf{X}_{i+1}\cdot\mathbf{B} =$$

$$(\mathbf{B}\cdot\mathbf{X}_{i+1}\times(\mathbf{A}_i\,\mathbf{i}))\,dx_i + (\mathbf{B}\cdot\mathbf{X}_{i+1}\times(\mathbf{A}_i\,\mathbf{j}))\,dy_i +$$

$$+ (\mathbf{B}\cdot(\mathbf{A}_{i+1}\,\mathbf{i})\times\mathbf{X}_i)\,dx_{i+1} + (\mathbf{B}\cdot(\mathbf{A}_{i+1}\,\mathbf{j})\times\mathbf{X}_i)\,dy_{i+1} \qquad (4.2)$$

The coefficients of the four corrections are scalar triple products of three vectors. Matrix multiplication shows that the vector $\mathbf{A}_i\,\mathbf{i}$ in the first coefficient is the vector whose components are the elements of the first column of $\mathbf{A}_i$ and that the vector $\mathbf{A}_i\,\mathbf{j}$ in the second coefficient is the vector whose components are the elements of the second column of $\mathbf{A}_i$. Analogous rules apply to $\mathbf{A}_{i+1}\,\mathbf{i}$ and to $\mathbf{A}_{i+1}\,\mathbf{j}$.

The weight of each equation (4.2) is a function of the four scalar triple products and of accuracy and correlation of the four photograph coordinates.

If no correlation exists between the coordinates and if all have the same accuracy, the weight of an equation is inversely proportional to the sum of the squares of the four scalar triple products. In the case of a strip of aerial photographs taken with an approximately vertical camera axis and of strip triangulation in the direction of the X-axis, the base components b_Y and b_Z will be small compared with b_X. Further, the x- and y-axes will be approximately parallel to the X- and Y-axes and, therefore, the diagonal elements of $\mathbf{A}_i$ and $\mathbf{A}_{i+1}$ will be approximately equal to unity and the off-diagonal elements will be small compared with unity. From this it follows that in each equation the coefficients of dx_i and dx_{i+1} will be approximately equal to zero and the coefficients of dy_i and dy_{i+1} will be approximately equal to $b_X f$. Consequently, in this case the weights are all approximately the same and may be made equal to unity.

In practice, the accuracy of the photograph coordinates depends upon the position of the points. An investigation of Hallert [20] gave the result that for a Wild RC8 camera approximately

$$m_x = m_y = k(1 + 7r^2), \tag{4.3}$$

where m_x and m_y are the standard deviations of the x- and y-coordinates, k is a constant, and r is the distance from the point to the principal point, expressed in the focal length as unit of length. If this result is accepted, each equation should be given the weight

$$w = \frac{1}{2 + 14(r_i^2+r_{i+1}^2) + 49(r_i^4+r_{i+1}^4)} \tag{4.4}$$

In this way, points near the principal points receive a weight that is about four times greater than the weight of points near the corners of a model.

5. Formation and solution of normal equations

Each point that is to be used to establish the relative orientation has now provided one linear equation (3.8). In the method of least squares, these equations are referred to as the correction equations. In matrix notation, they can together be represented by the equation

$$\mathbf{A}\,\mathbf{x} + \mathbf{b} = \mathbf{o} \tag{5.1}$$

Here, $\mathbf{A}$ is the matrix which has as the elements of each row the coefficients of one of the equations (3.8), $\mathbf{x}$ is the column vector whose components are the five unknowns and $\mathbf{b}$ is the column vector whose components are the constant terms. Obviously, these notations have no connection with the earlier used symbols.

If more than five correction equations are available, they are in general inconsistent and no vector $\mathbf{x}$ exists that can satisfy the matrix equation (5.1). According to the method of least squares, the most probable value of $\mathbf{x}$ is then the value for which the quadratic term

$$(\mathbf{A}\,x + \mathbf{b})^T\,\mathbf{W}\,(\mathbf{A}\,x + \mathbf{b})$$

attains its minimum. Under the present assumptions of uncorrelated observations, **W** is a matrix whose diagonal elements are the weights attached to the correction equations and whose off-diagonal elements are equal to zero.

It can be proved that the quadratic form attains its minimum for that value of x which satisfies the matrix equation

$$\mathbf{A}^T\,\mathbf{W}\,\mathbf{A}\,x = -\,\mathbf{A}^T\,\mathbf{W}\,\mathbf{b} \qquad (5.2)$$

This matrix equation comprises a set of five linear equations known as the normal equations.

The coefficients and the second parts of the normal equations could be computed by storing the complete matrix **A** and the vector **b** and by then performing the matrix multiplications $\mathbf{A}^T\mathbf{W}\,\mathbf{A}$ and $-\,\mathbf{A}^T\mathbf{W}\,\mathbf{b}$.

However, since here **W** is a diagonal matrix, the contribution of each correction equation to these matrix products can be computed separately. Let a correction equation be represented by the equation

$$a_r\,x + b = 0\ , \qquad (5.3)$$

where a_r is the row vector whose elements are the coefficients in the equation, x is again the column vector whose components are the five unknowns and b is the constant term. Let further a_c be the column vector which is the transpose of a_r. For each correction equation, a matrix $w\,a_c\,a_r$ and a column vector $-wb\,a_c$ can be computed. Here, w is the weight assigned to the equation. It can easily be shown that the matrix of coefficients and the vector of second parts in equation (5.2) are simply the sum of the matrices $w\,a_c\,a_r$ and the sum of the vectors $-wb\,a_c$, respectively. In this way, the normal equations are computed as:

$$[w\,a_c\,a_r] = [-wb\,a_c] \qquad (5.4)$$

The normal equations can be solved by Gaussian elimination and back substitution. In the elimination procedure, successive elements on the main diagonal of the matrix of coefficients can be used as pivotal elements.

6. Remarks on the derivations

i. If the components of the vector $\mathbf{r}$ in equation (3.5) are interpreted as infinitesimal rotations about the X_{i+1}, Y_{i+1}, and Z_{i+1} axes, the proof of equality of components given in equation (3.6) is not needed in the derivation of equation (3.8). According to a theorem from vector analysis, the three infinitesimal rotations change the vector $\mathbf{X}_{i+1}$ by vectors

$$a_1\,\mathbf{i} \times \mathbf{X}_{i+1},\ a_2\,\mathbf{j} \times \mathbf{X}_{i+1},\ \text{and}\ a_3\,\mathbf{k} \times \mathbf{X}_{i+1},$$

respectively. Therefore, that vector becomes $\mathbf{X}_{i+1} + r \times \mathbf{X}_{i+1}$. This expression can immediately replace the expression between the brackets in equation (3.1).

Although the three parameters have been defined as rotations, it is not necessary to interpret them as such when the orthogonal matrix $\mathbf{R}$ is computed. Instead, Table 5 can be used to change to a different set of parameters.

ii. A somewhat inelegant feature of the derivations is the use of matrix algebra (matrix-matrix and matrix-vector products) as well as vector algebra (dot products and cross products).

To avoid this, the equations can be written in a pure matrix notation. The vector r is then regarded as a column vector and the row vector which is its transpose is denoted by r^T. Its skew-symmetric matrix, $\mathbf{R}_1$ of equation (3.2), will be denoted by $[\,r\,]$. Equation (3.6) can now be written:

$$[\,r\,]\mathbf{X}_{i+1} = r \times \mathbf{X}_{i+1} \qquad (6.1)$$

Interchanging the two vectors, one has:

$$[\mathbf{X}_{i+1}]\, r = \mathbf{X}_{i+1} \times r = -\, r \times \mathbf{X}_{i+1} \qquad (6.2)$$

Following Thompson [21,22], the condition equation (2.4) can now be replaced by the equivalent expression

$$\mathbf{B}^T[\mathbf{X}_i]\mathbf{X}_{i+1} = 0 \qquad (6.3)$$

The introduction of approximate values of the orientation parameters and of corrections gives

$$(\mathbf{B}^T + d\mathbf{B}^T)[\mathbf{X}_i]\mathbf{R}\,\mathbf{A}_{i+1}\,x_{i+1} = 0 \qquad (6.4)$$

The linearization of this equation gives:

$$\mathbf{B}^T[\mathbf{X}_i][\,r\,]\mathbf{X}_{i+1} + d\mathbf{B}^T[\mathbf{X}_i]\mathbf{X}_{i+1} + \mathbf{B}^T[\mathbf{X}_i]\mathbf{X}_{i+1} = 0 \qquad (6.5)$$

$$-\mathbf{B}^T[\mathbf{X}_i][\mathbf{X}_{i+1}]\, r + ([\mathbf{X}_i]\mathbf{X}_{i+1})^T d\mathbf{B} + \mathbf{B}^T[\mathbf{X}_i]\mathbf{X}_{i+1} = 0 \qquad (6.6)$$

$$([\mathbf{X}_{i+1}][\mathbf{B}]\mathbf{X}_i)^T r + ([\mathbf{X}_i]\mathbf{X}_{i+1})^T d\mathbf{B} + \mathbf{B}^T[\mathbf{X}_i]\mathbf{X}_{i+1} = 0 \qquad (6.7)$$

Equation (6.7) is equation (3.8) in matrix formulation. For use in electronic computation, the operations on zero-elements of the skew-symmetric matrices should be omitted.

iii. A formulation purely in terms of vector algebra can be obtained if use is made of a third vector product, the dyad, and of the sum of such products, the dyadic. In the preceding sections of this chapter, these have not been used because they are relatively unknown. Here, they will be defined first and the required theorems will be given. From this formulation, an entirely analogous formulation in terms of three-dimensional tensor analysis can be obtained by replacing the vectors and dyadics by tensors of valence one and two, respectively.

The dyad $\mathbf{s}\,\mathbf{t}$, where $\mathbf{s}$ and $\mathbf{t}$ are vectors, is defined as an operator which transforms a third vector $\mathbf{u}$ into another vector as follows:

$$\mathbf{s}\,\mathbf{t}\cdot\mathbf{u} = \mathbf{s}\,(\mathbf{t}\cdot\mathbf{u})$$

and

$$\mathbf{u}\cdot\mathbf{s}\,\mathbf{t} = (\mathbf{u}\cdot\mathbf{s})\,\mathbf{t} \qquad (6.8)$$

Therefore, $\mathbf{s}\,\mathbf{t}\cdot\mathbf{u}$ is a vector which is parallel to $\mathbf{s}$ and which differs from $\mathbf{s}$ by the factor $\mathbf{t}\cdot\mathbf{u}$. Similarly, $\mathbf{u}\cdot\mathbf{s}\,\mathbf{t}$ is a vector parallel to $\mathbf{t}$. Because also $\mathbf{t}\,\mathbf{s}\cdot\mathbf{u}$ is a vector parallel to $\mathbf{t}$, the factors of the dyad do not commute.

Any dyad can be written as a sum of dyads (a dyadic) by writing one or both of its vectors as a sum of vectors and applying the distributive laws. This law applies also to the dot product of a sum of dyads and a vector. Thus, the products $\mathbf{A}\cdot\mathbf{u}$ and $\mathbf{u}\cdot\mathbf{A}$ of a dyadic $\mathbf{A}$ and a vector $\mathbf{u}$ can be written as a sum of vectors by first applying the distributive law and then using equations (6.8).

Each vector in a dyadic can be written as a linear function of the three unit vectors. As a result, using the distributive law and collecting the resulting term with the same dyad, any dyadic can be written in the form

$$\begin{aligned}\mathbf{A} = {} & a_{11}\mathbf{i}\,\mathbf{i} + a_{12}\mathbf{i}\,\mathbf{j} + a_{13}\mathbf{i}\,\mathbf{k} \\ & + a_{21}\mathbf{j}\,\mathbf{i} + a_{22}\mathbf{j}\,\mathbf{j} + a_{23}\mathbf{j}\,\mathbf{k} \\ & + a_{31}\mathbf{k}\,\mathbf{i} + a_{32}\mathbf{k}\,\mathbf{j} + a_{33}\mathbf{k}\,\mathbf{k} \qquad (6.9)\end{aligned}$$

The dyadic is then said to be in its nonion form, and the nine coefficients are called its components.

In the following, dyadics will always be used in their nonion form and vectors will be used in the form given in equation (3.5). If now the vector $\mathbf{u}$ has components u_1, u_2, and u_3, application of the distributive law gives:

$$\begin{aligned}\mathbf{A}\cdot\mathbf{u} = {} & (a_{11}u_1 + a_{12}u_2 + a_{13}u_3)\,\mathbf{i} \\ & + (a_{21}u_1 + a_{22}u_2 + a_{23}u_3)\,\mathbf{j} \\ & + (a_{31}u_1 + a_{32}u_2 + a_{33}u_3)\,\mathbf{k} \qquad (6.10)\end{aligned}$$

Accordingly, if A is the matrix formed by the nine components of the dyadic $\mathbf{A}$, the product $A\,u$ in the matrix formulation represents the same vector as the dyadic product $\mathbf{A}\cdot\mathbf{u}$. The matrix A is called the matrix of the dyadic $\mathbf{A}$.

The dot product of two dyadics $\mathbf{R}$ and $\mathbf{A}$ is the dyadic defined by the equation

$$(\mathbf{R} \cdot \mathbf{A}) \cdot x = \mathbf{R} \cdot (\mathbf{A} \cdot x) \quad (6.11)$$

It follows by substitution into this equation that the dot product of two dyads is

$$r s \cdot u v = (s \cdot u) r v \quad (6.12)$$

and that the matrix of the dyadic $\mathbf{R} \cdot \mathbf{A}$ is the product of the matrices $\mathbf{R}$ and $\mathbf{A}$.

The cross product of a dyad and a vector is defined by the equation

$$r s \times u = r (s \times u) \quad (6.13)$$

This is again a dyad. Through the distributive law, the definition is extended to the cross product of a dyadic and a vector.

It follows from the definitions that the following relation exists between any dyadic $\mathbf{A}$ and any two vectors r and s :

$$(\mathbf{A} \times r) \cdot s = \mathbf{A} \cdot (r \times s) \quad (6.14)$$

The unit dyadic or idemfactor is the dyadic whose matrix is the unit matrix:

$$\mathbf{I} = i i + j j + k k \quad (6.15)$$

From equations (3.5), (6.9), and (6.15) it follows easily that

$$\mathbf{I} \cdot r = r$$

$$\mathbf{I} \cdot \mathbf{A} = \mathbf{A} \quad (6.16)$$

Therefore, substitution of the idemfactor in equation (6.14) gives:

$$(\mathbf{I} \times r) \cdot s = r \times s \quad (6.17)$$

The cross product $\mathbf{I} \times r$ is in terms of the unit vectors:

$$\mathbf{I} \times r = a_1 (k j - j k) + a_2 (i k - k i) + a_3 (j i - i j) \quad (6.18)$$

Such a dyadic, whose matrix is skew-symmetric, is called anti-symmetric. Thus, equation (6.17) states the theorem that the cross product of two vectors is equal to the dot product of the anti-symmetric dyadic, formed from the first vector, and the second vector. This equation is the equivalent, in pure vector algebra, of equation (3.6).

With the help of the above equations, the condition equation (2.4) for relative orientation can be linearized. The equation itself remains unchanged. Dyadics $\mathbf{A}_{i+1}$ and $\mathbf{R}$ are now introduced to define the approximate orientation of photograph i+1 and its correction, respectively. The dyadics will be used in their nonion form, and their matrices will be defined as being the matrices A_{i+1} and R of the preceding sections.

This brings the condition equation in a form which is equivalent to equation (3.1):

$$B \cdot X_i \times (R \cdot (A_{i+1} \cdot x_{i+1})) = 0 \qquad (6.19)$$

Once R has been computed, the dyadic $R \cdot A_{i+1}$ will become the new approximation or the final value of A_{i+1}. The components of this dyadic are identical with the elements of the matrix $R A_{i+1}$.

The equation (6.19) is linearized by differentiation. The differentiation follows the ordinary rules for the differentiation of a product. The result is an equation which is equivalent to equation (3.4):

$$B \cdot X_i \times (R_1 \cdot X_{i+1}) + dB \cdot X_i \times X_{i+1} + B \cdot X_i \times X_{i+1} = 0 \qquad (6.20)$$

Here, R_1 is the antisymmetric dyadic formed from the parameters of R. Therefore

$$R_1 = I \times r \quad , \qquad (6.21)$$

where r is the vector of equation (3.5). By means of this equation and the theorem in equation (6.17), equation (6.20) can now be written

$$B \cdot X_i \times (r \times X_{i+1}) + dB \cdot X_i \times X_{i+1} + B \cdot X_i \times X_{i+1} = 0 \qquad (6.22)$$

This equation is identical with equation (3.7). It is converted to equation (3.8) by the method described in section 3.

7. Remarks on the computations

i. Choice of initial approximations

When a strip of aerial photographs is triangulated, the matrix A_i which is used in the correction equation for relative orientation can be the orientation matrix of photograph i computed during the relative orientation of that photograph. The best first approximation of the orientation matrix of photograph i+1 will then be the same matrix. This matrix was used in the NRC program for the IBM 650.

However, it is also possible to replace the matrix A_i in the correction equation by the unit matrix. The first approximation of the matrix A_{i+1} will then also be the unit matrix. After completion of the relative orientation, the obtained matrix A_{i+1} and the base B must then be premultiplied by the matrix A_i. This method has been used in the FORTRAN program.

Neither method has great advantages over the other. The change from one method to the other was made mainly because the

FORTRAN program was originally written for the triangulation of single models.

Because of the possibility of crab of the photographs, the best first approximation of the base component b_y is the final value of this component derived during the relative orientation of the preceding model.

ii. Simplification of the matrix **R**

In the FORTRAN program, equation (4.6) of chapter III is used to compute the correction matrix **R** from its three parameters.

If the division of each element of this matrix by their common denominator were omitted the matrix and, as a result, the transformed vectors $\mathbf{X}_{i+1}$ would be multiplied by the factor $1+a^2+b^2+c^2$.

Since a comparison of equations (3.2) and (4.6) in chapter III shows that the relations between the two sets of parameters in these equations are:

$$a = \lambda \tan \tfrac{1}{2} \alpha$$

$$b = \mu \tan \tfrac{1}{2} \alpha$$

$$c = \nu \tan \tfrac{1}{2} \alpha \qquad (7.1)$$

it follows that

$$1 + a^2 + b^2 + c^2 = 1 + \tan^2 \tfrac{1}{2} \alpha \qquad (7.2)$$

Even if the convergence of two photographs is 90° ($\alpha = 90°$, $b = \mu = 1$, $a = \lambda = c = \nu = 0$), this factor is only equal to 2. If the difference in tilt is 5°, it is equal to 1.002.

Therefore, if the division were omitted, the diagonal elements of the matrix **R** could become somewhat larger than unity. Because of the use of floating-point arithmetic, this will cause inaccurate values of the least significant digit of the elements of $\mathbf{A}_{i+1}$. If the length of the mantissas of the floating-point numbers is 10 decimal digits or more, this will have no effect upon the computed strip coordinates.

V. Absolute orientation and computation of strip coordinates

1. Absolute orientation

During the computation of the relative orientation, the base component b_X is assigned unit length.

The scaling of the model is now performed by computing the scale factor which reduces the model to the proper scale and by multiplying the three base components by it.

The first model of a strip can be given an arbitrary scale. This can be done by specifying the length of the base component b_X or of the base B . It is of advantage to specify that this length in the triangulated strip must be the same as the length in the photographs. In this case the triangulated strip will have the same scale as the photographs and the quality of the triangulation will immediately be evident from the size of the residual parallaxes and from the coordinate differences of points that are common to two models.

The scale of all other models must be reduced to that of the first model. For this purpose, one or more distances in the model are compared with the same distances in the preceding, already scaled, model. The ratio of the distances is accepted as the required scale factor and, therefore, as the numerical value of the base component b_X.

This computation presents no problem if a scale transfer point in the preceding model and the same point in the present model lie on the same ray through the projection centre of the common photograph. The required scale factor is then simply the ratio between the distances from the point to the common projection centre in the two models. This ratio is then equal to the ratio between the differences in Z-coordinates of the point and the projection centre in the two models.

However, in general the point will not lie on the same ray in the two models. This is a result of the facts, discussed in the next section, that after the adjustment of the relative orientation corresponding rays do not in general intersect and that the point that is defined as the point of intersection will not lie on either of them. In addition, if the readings have been made on a stereocomparator, the two sets of readings of the point in the common photograph may differ slightly and, therefore, may produce slightly different rays.

As a result, if a point lies at some distance from the axis of the strip, the two vectors from the common projection centre to a scale transfer point can have slightly different transversal tilts in the two models. In that case, the distances from the point to the projection centre and the heights of the point in the two models cannot simultaneously be the same. Therefore, scaling with these distances and scaling with these heights will give slightly different results.

Since want of intersection of corresponding rays is caused by uncorrected errors, the origin of these errors is presumably unknown and it is difficult to say which of the two methods of determining the scale will serve best to reduce the effect of these errors upon the triangulation.

In the above case, where the two sets of readings of the point in the common photograph differ slightly, the points measured in the two models are not precisely the same. It is then more likely that the two measured points have the same terrain height than that they are at the same distance from the common exposure station. Therefore, it is better to use the heights for the scaling than to use the distances to the common projection centre.

In the FORTRAN program, the scale factor is derived as the ratio between the distances from the points to the plane z = 0 of the common photograph. This is equivalent to using the heights of the points in a coordinate system in which the common photograph has tilts equal to zero. Therefore, the computed scale factor is independent of the tilt of the strip in the X, Y, Z coordinate system.

The distance from a point to the plane z = 0 is numerically equal to the z-component of the vector from the projection centre to the point. Therefore, if $\mathbf{A}_i$ is the definitive orientation matrix of the common photograph and the subscript P refers to the common projection centre, these distances d_1 and d_2 in the preceding and the present model are:

$$d_1 = (\mathbf{X} - \mathbf{X}_P) \cdot (\mathbf{A}_i \mathbf{k})$$

$$d_2 = (\mathbf{X} - \mathbf{X}_P) \cdot \mathbf{k} \qquad (1.1)$$

Here, $\mathbf{X}$ in the first equation is the definitive position vector of the point in the preceding model and $\mathbf{X}$ in the second equation is the position vector in the present model before absolute orientation. The second equation is simpler than the first one because in the present model the orientation matrix of the common photograph is still the unit matrix.

The scale factor is now computed as the mean of the ratios d_1/d_2 for all scale transfer points, and the base vector is multiplied by this factor.

The absolute orientation is completed by pre-multiplying the base and the orientation matrix of the new photograph by the definitive orientation matrix of the common photograph:

$$\mathbf{B} = (d_1/d_2)_{mean} \mathbf{A}_i (\mathbf{B})_{rel.\ or.}$$

$$\mathbf{A}_{i+1} = \mathbf{A}_i (\mathbf{A}_{i+1})_{rel.\ or.} \qquad (1.2)$$

The position vector of the projection centre Q of the new photograph is computed by adding the base vector to the position vector of the projection centre P of the common photograph:

$$\mathbf{X}_Q = \mathbf{X}_P + \mathbf{B} \qquad (1.3)$$

2. The point of intersection of two rays

The strip coordinates of all measured points can now be computed by intersecting the rays from corresponding image points in the two photographs.

If the photograph coordinates are free from all errors, the relative orientation is also error-free and each two such rays intersect. In practice that is not the case and consequently the rays cross at a short distance.

This makes it necessary to select a point that is to represent the point of intersection.

There are three points that can be used as such. The selection that is made will depend upon the point of view.

i. If the use of the method of least squares in the adjustment of the relative orientation is considered to be the best procedure, and not only a convenient one, it is logical to use this method also to give the photograph coordinates such corrections that each two rays do intersect. The point of intersection of the corrected rays will then be used.

This adjustment can be based upon equation (4.2) in chapter IV and it can be applied not only to the points which have been used to establish the relative orientation but also to all other points. Since orientation corrections are not computed at this stage, the terms with r and with dB are omitted from the equation. Representing further the constant term and the coefficients of the remaining terms by d, d_1, d_2, d_3, and d_4, the equation becomes

$$d_1 dx_i + d_2 dy_i + d_3 dx_{i+1} + d_4 dy_{i+1} = d \tag{2.1}$$

Assuming equal accuracy of the coordinates and freedom from correlation, the application of the method of least squares to this equation gives the following corrections to the photograph coordinates:

$$\begin{aligned} dx_i &= d_1 d/(d_1^2 + d_2^2 + d_3^2 + d_4^2) \\ dy_i &= d_2 d/(d_1^2 + d_2^2 + d_3^2 + d_4^2) \\ dx_{i+1} &= d_3 d/(d_1^2 + d_2^2 + d_3^2 + d_4^2) \\ dy_{i+1} &= d_4 d/(d_1^2 + d_2^2 + d_3^2 + d_4^2) \end{aligned} \tag{2.2}$$

ii. If the strip triangulation is performed on a first-order plotter, the point which represents the point of intersection is defined as a point in the horizontal plane at the height where the X-parallax between the two rays is equal to zero. It lies on the line which connects the points of intersection of the rays and this plane, midway between these points.

This point is obtained also if the first method of correcting the photograph coordinates is applied to exactly vertical photographs. This is the result of the fact that in this case, with the x_i- and x_{i+1}-axes parallel to the X-axis, the coefficients d_1 and d_3 become equal to zero while the coefficients d_2 and d_4 become equal. Since these coefficients change very little when the photographs are given small tilts, this choice of point is suitable for all nearly vertical photographs.

iii. If one wishes to represent the point of intersection by the point that lies as close to the two rays as possible, the point which lies midway on the line of shortest distance of the non-intersecting rays should be selected.

If the photographs are of a good quality the want of intersection of corresponding rays is small and consequently the three points are almost identical. If systematic errors occur, the method of least squares should be looked upon as only a convenient adjustment procedure. It is then impossible to say which point represents the point of intersection best. Therefore, in practice each of the three points is acceptable.

In the NRC program for the IBM 650, the second choice was made. In the present FORTRAN program, the third point has been chosen. However, a return to the second point would only require changing a few FORTRAN statements.

3. Computation of the point of intersection

In case ii and in case iii of the preceding section, the "point of intersection" of two corresponding rays lies on a specified line from a point on one of the rays to a point on the other ray, and midway between those two points.

This situation is shown in Figure 8, in which $\mathbf{X}_i$ and $\mathbf{X}_{i+1}$ are the vectors from the projection centres to the image points in the oriented photographs and $\mathbf{D}$ is a vector of arbitrary length,

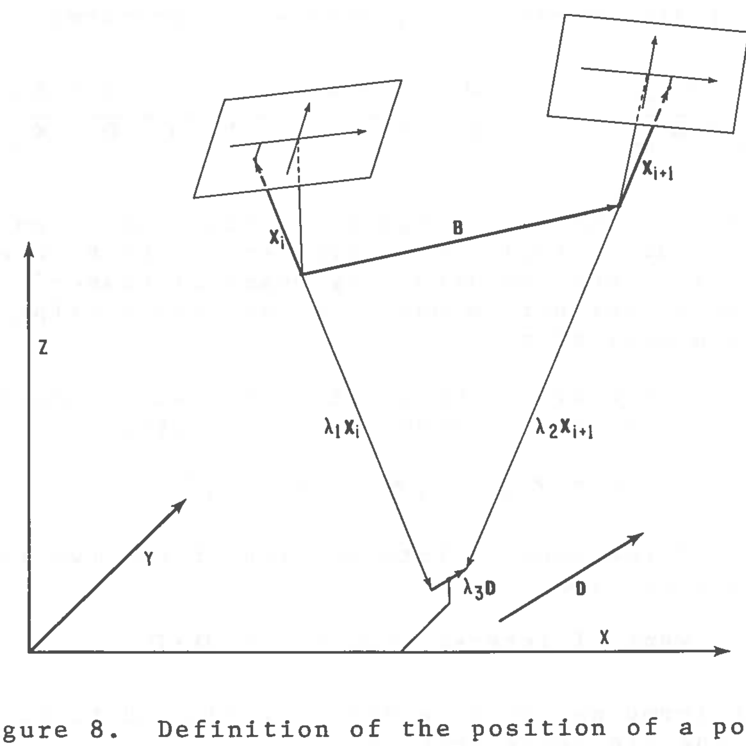

Figure 8. Definition of the position of a point in the case of non-intersecting rays.

parallel to the line on which the point of intersection lies.

The two vectors from the projection centres to the above two points are collinear with $\mathbf{X}_i$ and $\mathbf{X}_{i+1}$ and, therefore, can be denoted by $\lambda_1 \mathbf{X}_i$ and $\lambda_2 \mathbf{X}_{i+1}$, in which the factors λ_1 and λ_2 are scalars. Similarly, the vector which connects the two points is parallel to $\mathbf{D}$ and can be denoted by $\lambda_3 \mathbf{D}$.

It follows directly from Figure 8 that

$$\mathbf{B} = \lambda_1 \mathbf{X}_i - \lambda_2 \mathbf{X}_{i+1} + \lambda_3 \mathbf{D} \tag{3.1}$$

According to a theorem from vector analysis, any four vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ in three-dimensional space are connected by the equation

$$(\mathbf{a} \cdot \mathbf{b} \times \mathbf{c})\mathbf{d} - (\mathbf{b} \cdot \mathbf{c} \times \mathbf{d})\mathbf{a} + (\mathbf{c} \cdot \mathbf{d} \times \mathbf{a})\mathbf{b} - (\mathbf{d} \cdot \mathbf{a} \times \mathbf{b})\mathbf{c} = 0 \tag{3.2}$$

This theorem is derived by regarding a vector product of these vectors in two ways as a triple vector product, expanding it and equating the two results:

$$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \times \mathbf{b} \cdot \mathbf{d})\mathbf{c} - (\mathbf{a} \times \mathbf{b} \cdot \mathbf{c})\mathbf{d}$$

$$= (\mathbf{a} \cdot \mathbf{c} \times \mathbf{d})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c} \times \mathbf{d})\mathbf{a} \tag{3.3}$$

Since equation (3.1) is valid only for unique values of λ_1, λ_2, and λ_3, and equation (3.2) is valid for any four vectors, the values of λ_1, λ_2, and λ_3 follow immediately when the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ are equated with the vectors $\mathbf{B}$, $\mathbf{X}_i$, $\mathbf{X}_{i+1}$, and $\mathbf{D}$, and the coefficients of the two equations are compared. This gives

$$\lambda_1 = \frac{\mathbf{D} \cdot \mathbf{B} \times \mathbf{X}_{i+1}}{\mathbf{D} \cdot \mathbf{X}_i \times \mathbf{X}_{i+1}}, \quad \lambda_2 = \frac{\mathbf{D} \cdot \mathbf{B} \times \mathbf{X}_i}{\mathbf{D} \cdot \mathbf{X}_i \times \mathbf{X}_{i+1}}, \quad \lambda_3 = \frac{\mathbf{B} \cdot \mathbf{X}_i \times \mathbf{X}_{i+1}}{\mathbf{D} \cdot \mathbf{X}_i \times \mathbf{X}_{i+1}} \tag{3.4}$$

Alternatively, but less elegantly, this result can be obtained by decomposing equation (3.1) into three equations between vector components, solving these equations by means of Cramer's rule and replacing the resulting determinants by the scalar triple products to which they are equivalent.

The position vector of the point in the strip coordinate system can now be computed by means of the equation

$$\mathbf{X} = \mathbf{X}_P + \lambda_1 \mathbf{X}_i + 0.5\,\lambda_3 \mathbf{D} \tag{3.5}$$

A measure of the want of intersection of the two rays is the length of the vector $\lambda_3 \mathbf{D}$:

$$\text{Want of intersection} = \lambda_3 \sqrt{(\mathbf{D} \cdot \mathbf{D})} \tag{3.6}$$

The above formulas can be especially adapted to each of the three cases in the preceding section.

In case i, the rays intersect after correction of the photograph coordinates. As a result, λ_3 is equal to zero and any vector which is not parallel to the plane which contains the base and the corrected image points can serve as the vector **D**. A simple set of equations is obtained by taking for **D** the unit vector **j** in the positive Y-direction.

In case ii, the vector **D** must be parallel to the Y-axis. Here also, the unit vector **j** in the positive Y-direction is a suitable choice. This gives for λ_1 and λ_2 the simple expressions

$$\lambda_1 = \frac{b_Z X_{i+1} - b_X Z_{i+1}}{Z_i X_{i+1} - X_i Z_{i+1}} \quad \text{and} \quad \lambda_2 = \frac{b_Z X_i - b_X Z_i}{Z_i X_{i+1} - X_i Z_{i+1}} \tag{3.7}$$

The X- and Z-coordinates of the point are now computed as the X- and Z-components of the vector $\mathbf{X}_P + \lambda_1 \mathbf{X}_i$ or, which gives the same result, those of the vector $\mathbf{X}_Q + \lambda_2 \mathbf{X}_{i+1}$. The Y-coordinate is computed as the mean of the Y-components of these two vectors and the want of intersection is computed as the difference.

In the case iii, the vector **D** must be parallel to the line of shortest distance. The simplest choice, and the one used in the FORTRAN program, is

$$\mathbf{D} = \mathbf{X}_i \times \mathbf{X}_{i+1} \tag{3.8}$$

This vector product is here substituted for **D** in the equations (3.4), (3.5), and (3.6). The want of intersection is positive if, in the area where the rays cross, points on the ray from photograph i+1 have greater Y-coordinates than those on the ray from photograph i, and it is negative if the former points have smaller Y-coordinates.

The computation of the position vectors of all measured points and of the want of intersection completes the computations for the model.

VI. A FORTRAN IV Program

1. General Remarks

The present version of the program employs the mathematical formulation described in the preceding chapters. In addition, it contains the following features:

i. The photograph coordinates can be corrected for differential film shrinkage.

ii. The corrections for symmetric lens distortion are derived from a table of radial corrections for points spaced at equal intervals along a radius.

iii. The radial corrections for earth curvature and refraction are applied directly by the program rather than being included in the table.

iv. The relative orientation of each photograph is started with the unit matrix as the orientation matrix of the preceding photograph and with the base component b_Z equal to zero.

v. The elements of the orientation matrix are computed as rational functions of parameters.

vi. For relative orientation, any number of points may be used. The number of iterations is controlled by test values.

vii. An experimental formula for weighting the equations for relative orientation is included as an option.

viii. The point of intersection of corresponding rays is defined as the point midway on their line of shortest distance. The want of intersection of corresponding rays is defined as their shortest distance.

ix. Points for scaling a model must be selected from the points used for relative orientation. These points are given equal weights.

x. The scaling is performed by comparing for each scale transfer point its distance in the present and in the preceding model to a plane which contains the common projection centre and is parallel to the plane of the common photograph.

2. The iterative procedure of relative orientation

In chapter IV, the condition equation for relative orientation has been linearized and it has been shown how the linear equation can be used in an iterative procedure to determine the relative orientation.

In the first iteration, the unit matrix is used as the matrix of the approximate orientation of photograph i+1. The base component b_Y is made equal to the final value computed during the relative orientation of the preceding model or, in the case of the first model of a strip, zero.

From two to four iterations of the relative orientation are performed. After each iteration, the computed matrix parameters and base ratios are compared with a test value (1/30th of a radian).

Once they are all smaller than this value it is certain that the corrections which further iterations will apply will be of the order of 10' or smaller. Such corrections can be obtained by one iteration and, accordingly, only one more iteration is then performed.

The test value (30.) is contained in the second FORTRAN statement preceding the one labelled 1332. The variable ITMAX controls the maximum number of iterations that is performed.

In each iteration except the final one, the FORTRAN program uses only those points whose coordinates can be stored in the arrays in core storage (in the listed program at the most 35; see the arrays LIST, LYST, SLIST, and PHC and the constant K4MAX) and it applies the same weight to all these points. In the final iteration, there is no restriction on the number of points that can be used. Here, either the contribution of each correction equation to the normal equations receives the same weight or its weight is computed by means of equation (4.4) of chapter IV. The weight function can be modified by changing the constants 14. and 49. in the second statement following the one labelled 1252.

3. The scaling of models

The scale of the triangulated strip is determined by the value of the base component b_X of the first model. This value is punched in the first card of the input deck for the strip and is expressed in micrometers at photograph scale. Therefore, if this value is equal to the actual length at photograph scale and the base components b_Y and b_Z are small, the computed strip coordinates and parallaxes will be expressed also, at least approximately, in micrometers at photograph scale.

For the scaling of the first model, only the base component b_X is used. Each following model is scaled to the preceding one by means of points in the overlap of the two models. In each of these two models, each scale transfer point must occur among the relative orientation points that can be stored in the arrays in core storage. The points are given equal weights.

Each scale transfer point must have the same point number in the two models in which it occurs. If a point occurs in two successive models, but must not be used as a scale transfer point, a minus sign must be attached to the point number in one or both models. If a point, or a point number, occurs in three successive models, the point will not be recognized as a scale transfer point for the third model.

The program searches for anomalous scale transfer points and discards these one at a time. A scale transfer point will be discarded if the difference between the scale factor derived from it and the mean of the scale factors from all as yet not discarded points is larger than 0.0008 times this mean. As shown by the results of triangulations performed with the data of the Working Group on Analytic Block Adjustment of Commission III, I.S.P., 1972, this test value is suitable for average film photography. It can be changed by punching an optional value in the first card of the data deck.

4. Input

The input data formats have been designed for the use of cards.

Each deck of cards for the triangulation of a strip must contain all the information that is needed to perform the triangulation. Decks for different strips may be stacked for sequential processing. In this case, they must be separated from each other by means of a card with a negative non-zero number in field 1 (col. 1-8). A blank card must be placed behind the deck for a single strip triangulation and behind the last deck of a stack.

All data punched in the cards must be in the form required by FORTRAN fixed-point format: the least significant digit of each number must be punched in the right-most column of its field, and if a number is negative a minus sign must be punched in one of the columns of its field to the left of the most significant digit.

i. Cards with general information

The first few cards in the deck of a strip contain necessary data other than the measured coordinates.

The first card contains first four four-digit fields which serve to introduce options. To exercise an option, an integer must be punched in the relevant field. These fields and the corresponding options are:

field 1A, col. 1-4: A non-zero integer in this field causes the continuous strip triangulation to be replaced by a triangulation of independent models. The data in the card or cards with general information is used for each model.

field 1B, col. 5-8: A non-zero integer in this field causes card output to be suppressed.

field 2A, col. 9-12: A non-zero integer in this field causes the correction equations for relative orientation to be weighted according to the experimental formula.

field 2B, col. 13-16: In the case of a continuous strip triangulation, a positive number punched in this field will, after division by 10000, replace the test value 0.0008 for discarding anomalous scale transfer points.

The next six eight-digit fields in the first card contain the following information, punched as integers.

field 3, col. 17-24: The calibrated focal length in micrometers.

field 4, col. 25-32 and
field 5, col. 33-40: Correction factors for average film shrinkage in x- and y-direction, respectively, multiplied by 100000.

field 6, col. 41-48: The value of the base component b_X of the first model, expressed in micrometers at photograph scale.

field 7, col. 49-56: The average flying height above ground, in meters. This value will be used for the computation of the earth curvature correction. If this correction must not be applied, this field must contain zeros or blanks.

field 8, col. 57-64: The coefficient c_1 in the formula for the refraction correction, multiplied by 10^7. This is the value of the photogrammetric refraction found by interpolation in Table II for the actual flying height and the average terrain height, multiplied by 10. It must be punched as a positive integer. If this correction must not be applied, this field must contain zeros or blanks.

Columns 65-76 in the first card are not referred to by the program. The last four-digit field serves to introduce the symmetric radial correction for lens distortion as an option, as follows:

field 10B, col. 77-80: A positive integer punched in this field signifies that corrections for symmetric radial lens distortion shall be applied.

If and only if the first card specifies that corrections for lens distortion shall be applied, this card must be followed by one or more cards with the data for these corrections. This data consists of a list of values of the radial lens correction dr for values of the radial distance r which are separated by a constant interval and start at r = 0. The interval must be sufficiently small to allow linear interpolation between consecutive table values. The last value of the lens correction must apply to a radial distance which is larger than the distance from the principal point to a corner of the photograph.

The first lens correction card contains:

field 1A: The number of entries in the table. This number must not be larger than 160.

field 1B: The interval of the argument r, expressed in 0.1 mm as unit of length.

fields 2 through 9, covering columns 9 through 72:

Up to eight values of the lens correction dr, starting with dr = 0 for r = 0, and followed by the values for consecutive values of r. The lens corrections must be expressed in 0.1 micrometer as unit of length.

field 10B: The serial number of the card. This is the integer that is one larger than the integer in this field of the preceding card.

If the lens correction data contains more than eight items, the remainder are punched sequentially in fields 2 through 9 of the next card or cards. Each following card must contain also a serial number that is one higher than the one in the preceding card. The serial number is used by the program to check the proper sequence of these cards in the data deck.

Columns 73-76 of the lens correction cards are not referred to by the program.

ii. Cards with comparator coordinates

The remaining cards contain the comparator coordinates. They are arranged in groups according to the models, and the models are arranged in the sequence in which the triangulation is to be performed.

The first card of a model contains the coordinates of the principal points of the two photographs. Each of the following cards contains the coordinates of corresponding image points in the two photographs. The cards of the points that are to be used for relative orientation come first and are followed by the cards of any additional points.

The cards are punched in 8-digit fields as follows:

field 1: For cards of image points, the model identification (numeric, positive, and the same for all image points in a model).

For cards of principal points, any positive number is acceptable except the model identification of the preceding model. For the first principal point card in a strip, the number of the first photograph should be punched in this field.

field 2: Point identification (numeric). For a principal point card this should be the number of the second photograph in the model.

field 3: x-coordinate in the oriented photograph,

field 4: y-coordinate in this photograph,

field 5: x-coordinate in the new photograph,

field 6: y-coordinate in this photograph; all in micrometers.

In addition, the principal point card contains a 4-digit field:

field 7A (columns 49-52) of the principal point card: The number of points to be used for relative orientation.

As an example of an assembled data deck, the listing for two models of a strip is shown in Table 6. The photography was taken with a 6" focal length camera at a height of about 700 m above sea level over terrain with an average elevation of about 300 m. A continuous strip triangulation is specified, with corrections for earth curvature, refraction, and lens distortion.

Table 6. Example of Input

		0005	152740	100000	99930	88000	400	51	SUDB.66	RC8	1
51	30	0	-13	-25	-35	-45	-54	-62	-70	RC8	2
		-78	-85	-91	-97	-102	-106	-109	-111	RC8	3
		-112	-111	-109	-105	-101	-96	-89	-82	RC8	4
		-74	-65	-57	-47	-37	-26	-15	-2	RC8	5
		11	24	37	49	61	72	82	89	RC8	6
		94	93	89	83	75	65	55	43	RC8	7
		30	15	0						RC8	8
	69	70	120343	118614	119715	118943	10				
	5070	1001	120523	223974	36397	223122					
	5070	1002	132629	122445	47553	121117					
	5070	1003	93605	124372	10981	123073					
	5070	1004	100176	14086	22127	17638					
	5070	1010	152235	17789	69085	20341					
	5070	2004	199052	18650	113297	20487					
	5070	2003	171382	127079	84606	125669					
	5070	2002	218270	123049	132377	121827					
	5070	2001	200915	222512	116082	222691					
	5070	1009	150244	227139	65455	226770					
	5070	149	100177	14084	22127	17638					
	5070	151	119773	66341	34725	66653					
	5070	31	198200	105289	109110	104017					
	5070	185	139233	129921	52923	128414					
	5070	16	91349	156199	7608	154360					
	5070	184	211999	186798	126626	186079					
	70	71	119715	118943	120236	119416	10				
	5071	2001	116082	222691	49782	215326					
	5071	2002	132377	121827	59147	116384					
	5071	2003	84606	125669	12999	123122					
	5071	2004	113297	20487	35380	18275					
	5071	2010	152643	26698	72925	21080					
	5071	3004	202243	30954	120437	21106					
	5071	3002	221938	132637	145954	121191					
	5071	3003	178761	124887	103461	116349					
	5071	3001	193339	225947	122988	214860					
	5071	2009	149634	220311	81151	211450					
	5071	31	109110	104017	33171	100400					
	5071	43	166152	51144	87120	44227					
	5071	183	154512	122135	79341	115189					
	5071	184	126626	186079	57328	179343					
	5071	32	215611	181117	141274	169573					
	0										

5. Restrictions on the input data

The x-axis must be chosen roughly in the direction of the strip. For the purpose of the triangulation, it is not necessary to place the photographs in the comparator with the lines which connect the fiducial marks exactly parallel to the coordinate axes. This is necessary only if further use of the orientation matrices or their parameters is envisaged. If a continuous strip triangulation is to be performed, the position of a photograph must not be changed between the measurements for the two models in which it participates.

The positive direction of the x-axis may be chosen at will, but it must be the same for all photographs of a strip.

The specifications for card output make it necessary to restrict the model identification to at the most four digits and the point identification to at the most six digits.

The numbers in fields 1 of the coordinate cards are used to recognize the cards of all image points that belong to the same model. Therefore, these numbers must be the same for all image points of the same model and they must be different for different models. For the same reason, the number in field 1 of the principal point card may be the same number as the one in field 1 of the image cards of the same model but it must not be the same as the number in field 1 of the image cards of the preceding model.

At least six points must be used for the relative orientation of a photograph. Five suitably located points would be sufficient to define the relative orientation; a sixth point is needed as a check on errors.

There is no upper limit to the number of points that may be used for relative orientation of a photograph, but the strip coordinates and parallaxes of only those relative orientation points that can be accommodated in the coordinate arrays in storage will be printed and punched. Those of any excess orientation points can be obtained by punching for each a second card with the measured coordinates and placing these cards behind the cards for relative orientation.

6. Output

The program produces listed output and card output. The card output is needed only for subsequent strip- and block-adjustment and can be suppressed.

Model by model, the following lines are printed.

i. One line for each iteration of the relative orientation that is performed. These lines contain in field 1 the number of the iteration. The following fields contain the corrections a_1, a_2, a_3, db_Y, and db_Z.

ii. One line for each scale transfer point that is rejected because of an anomalous scale factor. This line contains the point number of the rejected point, with a minus sign attached.

iii. Three lines for the orientation matrix of the new photograph. These lines contain the model number and the elements of the matrix, row by row.

iv. One line for the projection centre of the first photograph, but only for the first model of a strip and for independent models.

v. One line for the projection centre of the new photograph.

vi. One line for each of the measured points.

These lines contain:

field 1: Model number.

field 2: Point number. In the case of a continuous triangulation, a minus sign is added to the point number if it occurs also in the preceding model but the point has been rejected for scale transfer for one of the following reasons:

a. In the input data, a minus sign is attached to the point number in one or both of the two models.

b. The point contributes an anomalous scale factor.

c. The point number occurs in the two preceding models and the point has been rejected in the preceding model.

fields 3, 4, and 5: X-, Y-, and Z-coordinates in the model.

For terrain points, they contain also:

field 6: The want of intersection, that is, the minimum distance between corresponding rays.

fields 7, 8, and 9: The differences in X-, Y-, and Z-coordinates for both used and unused tie points between models.

If card output is produced, identification and strip coordinates are punched one point per card. Here, all point numbers are positive. For each projection centre, the parameters of the orientation matrix of the photograph are punched also. These are punched as integers, after multiplication by 10^7. For each used scale transfer point, only one card is produced. This card contains the means of the coordinates of the point obtained from the two models in which it occurs. For unused scale transfer points, a card is punched for each model in which they occur.

The program produces the card output in 10-digit fields. If the output is to be used as input to the strip and block adjustment program in AP-PR 33, either the output format specified in the FORTRAN statement with label 5 must be modified to suit the requirements of that program or the read format of that program must be modified.

The listing obtained from the input data in Table 6 is shown in Table 7. Differences from the output listing in publication AP-PR 34 of December 1966 are caused by the use of a more accurately interpolated value of the refraction coefficient and by the introduction of one coordinate error of 100 micrometers.

7. Error detection

The program contains a number of tests to detect errors in the data deck. If an error is detected, an error message is printed. The following error messages can occur:

ERROR 1 YYYY The first card in the data deck of a strip specifies that corrections for lens distortion shall be given, but either the cards with values of the lens correction are absent or they are not in the correct sequence or there are more values than the allowed maximum. YYYY is the contents of field 10B of the last card that has been read. After this message, the triangulation of the strip is discontinued and any remaining cards of the strip are bypassed.

Table 7. Example of Listed Output

1	-0.03272628	0.01800009	-0.00167139	0.07674471	0.01631291			
2	0.00006797	0.00278193	0.00098578	-0.00237810	-0.00339969			
3	0.00000159	-0.00000108	-0.00000054	0.00001157	0.00001547			
5070	0.99978428	0.00030073	0.02076779					
5070	-0.00097884	0.99946639	0.03264941					
5070	-0.02074689	-0.03266269	0.99925108					
5070	69	200000	400000	600000				
5070	70	288000	406545	601137				
5070	1001	200183	507268	444393	1			
5070	1002	212152	403783	448861	-6			
5070	1003	173168	405772	446674	-4			
5070	1004	179140	291958	442026	-1			
5070	1010	231994	298924	446772	11			
5070	2004	278155	300804	448350	-10			
5070	2003	250039	408294	450217	1			
5070	2002	297497	404417	447933	10			
5070	2001	281259	504707	445966	-8			
5070	1009	230293	509875	445268	6			
5070	149	179142	291959	442028	1			
5070	151	199445	349161	451318	-2			
5070	31	274621	387238	453597	2			
5070	185	218482	411055	450501	-1			
5070	16	171050	437497	447455	-9			
5070	184	291733	468201	447144	15			
1	0.01083650	0.03030139	0.06587031	0.03326883	0.01430414			
2	-0.00163624	-0.00006741	-0.00051061	0.00271839	-0.00138228			
3	-0.00000057	0.00000502	-0.00000195	-0.00000199	-0.00000156			
-2004								
5071	0.99659303	-0.06458165	0.05129852					
5071	0.06339665	0.99769002	0.02440253					
5071	-0.05275597	-0.02106724	0.99838519					
5071	71	367428	415261	600229				
5071	2001	281260	504690	445991	5	1	-17	25
5071	2002	297497	404420	447929	-3	0	3	-4
5071	2003	250035	408290	450197	-9	-4	-4	-20
5071	-2004	278171	300983	448609	9	16	179	259
5071	2010	317941	308671	449731	-7			
5071	3004	366155	314951	451893	-3			
5071	3002	384731	414938	449557	4			
5071	3003	343139	407429	449058	9			
5071	3001	356366	505317	448204	-8			
5071	2009	314373	501345	446915	3			
5071	31	274625	387236	453633	-15			
5071	43	331177	333887	449960	-17			
5071	183	319129	404735	450010	-8			
5071	184	291732	468201	447162	-1			
5071	32	376982	461255	450822	-15			

ERROR 2 YYYY The distance from an image point to a principal point exceeds the largest distance for which a lens correction has been specified. YYYY is the point identification. After this message the triangulation is continued but no corrections are applied to the photograph coordinates of the point.

ERROR 3 YYYY Either a principal point card specifies that fewer than six points shall be used for relative orientation or there are fewer points in the model than the specified number. YYYY is the specified or the available number, respectively. If fewer than six points are available, the triangulation is discontinued and the remaining cards of the strip are by-passed. Otherwise, the triangulation is continued using the available points.

ERROR 4 YYYY A continuous strip triangulation shall be performed, but no points are available to scale the present model with respect to the preceding one. YYYY is the number of the present model. After this message, a new triangulation is started, beginning with the present model, and any further punching is suppressed.

The following two errors do not produce an error message.

i. The program has drawn the wrong conclusion concerning whether the photographs are in positive or in negative position.

This conclusion is based upon the sign of the difference of the x-coordinates of the first orientation point, after shifting the origins to the principal points, and upon the assumption that a positive value of the focal length has been specified on the first card in the data deck. If the convergence of the camera axes is very great, the sign of the x-difference can differ from the sign in the case of parallel axes. If this is the case for the first orientation point, the model that is computed will be upside down. The correct result can then be obtained either by specifying a negative value of the focal length or by placing the card of a suitable point directly behind the principal point card.

ii. An error occurs in one of the coordinates punched in the input cards.

The program can detect such an error only if it either occurs in a scale transfer point and affects the height of the point or results in an ERROR 2 message. In the first case, the affected point will still be used for relative orientation but it will not be used for scaling. If the point shows only a small want of intersection, the error will not appreciably affect the triangulation.

8. Symbols in the FORTRAN statements

AL: Array for the orientation matrix $\mathbf{A}_i$ of the first photograph of a model (column after column).

AR: Array for the orientation matrix $\mathbf{A}_{i+1}$ of the second photograph of a model.

A1, A2, A3, DBY, DBZ:	The parameters of relative orientation.
BX, BY, BZ:	The three base components.
CLIST:	Array for the corrections for lens distortion.
DELR:	The interval of the radial distances for which the lens correction is listed.
F, CX, CY, BX1, CE, CR:	The quantities in fields 5 to 10 of the first data card.
ID, ID1, ID2, ID3:	Locations for storing model numbers.
IPC, IPC1, IPC2:	Locations for storing photograph numbers.
IX, IY, IZ:	Strip coordinates.
K, KA, KC, KD, MOD:	Counters which control the course of the computations. In particular, K progresses from 1 to 3 during the computation of each model. MOD is equal to 1 for the first model and is equal to 2 for all following models.
K1:	Counter for the points used for relative orientation.
K2:	Index which specifies whether or not corrections for lens distortion are applied.
K3:	Index which specifies whether the photographs are in positive or in negative position.
K4:	Index which specifies the number of points to be used for relative orientation (temporary storage is in K6).
K4MAX:	Index which states how many of these points can be accommodated in arrays in storage.
K5:	Index which is equal to the smaller one of K4 and K4MAX.
K8:	Index used for error messages.
LIST and LYST:	Arrays for the point numbers and strip coordinates of the points used for relative orientation.
PHC:	Array for the photograph coordinates of the points used for the relative orientation.
PP:	Array for the photograph coordinates of the principal points.

PX, PY, PZ:	The strip coordinates of the projection centre of the first photograph of a model.
R:	Array for the matrix **R** for correction of the relative orientation.
S:	Array for the normal equations.
SLIST:	Array for the distances and the scale factors computed for each scale transfer point.
T, and T1 to T9:	Array for the correction equations, and its elements. Also used for temporary storage.
W, and W1 to W4:	Array for temporary storage of the photograph coordinates of a point in first and second photograph of a model, and its elements.
WANT:	The want of intersection.
XL and X1, Y1, Z1:	Array for the components of the vector $\mathbf{X}_i$ of the point in the first photograph of a model, and its elements.
XR and X2, Y2, Z2:	Array for the components of the vector $\mathbf{X}_{i+1}$ of the point in the second photograph, and its elements.

9. Listing of the FORTRAN statements

```
C       ANALYTICAL STRIP TRIANGULATION.
C       N.R.C. PROGRAM OF MARCH 1973 - G.H.S.
C
        DIMENSION LIST(4,36), LYST(5,35), CLIST(160)
        DOUBLE PRECISION T(30),R(3,3),AR(3,4),AL(3,3),XL(3),
     1      XR(3),SLIST(35),PHC(4, 36),W(4),PP(4),S(5,6)
        DOUBLE PRECISION T1,T2,T3,T4,T5,T6,T7,T8,T9,
     1      X1,Y1,Z1, X2,Y2,Z2, W1,W2,W3,W4, BX,BY,BZ,
     2      A1,A2,A3,DBY,DBZ, PX,PY,PZ, DELR,WANT,
     3      F,CX,CY,BX1,CE,CR, BY1,FOCAL, RMAX
        COMMON T1,T2,T3,T4,T5,T6,T7,T8,T9, AR,AL,
     1      X1,Y1,Z1, X2,Y2,Z2, W1,W2,W3,W4
        EQUIVALENCE (T1,T(1),R(1,1)), (W1,W(1)),
     1 (BX,AR(1,4)), (BY,AR(2,4)), (BZ,AR(3,4)),
     2 (A1,S(1,6)), (A2,S(2,6)), (A3,S(3,6)),
     3 (DBY,S(4,6)), (DBZ,S(5,6)), (X1,XL(1)), (X2,XR(1))
      1 FORMAT  (3I4,F4.4, F8.3,2F8.5,F8.0,F8.3,F8.7, 12X, I4)
      2 FORMAT  (I4,F4.1, 8F8.1, 4X, I4)
      3 FORMAT  (2I8, 4F8.3, I4)
      4 FORMAT  (1H I9, 3X, 5F13.8)
      5 FORMAT  (I4, I6, 7I10)
     55 FORMAT  (1H 7I9, 2I7)
      6 FORMAT  (6H0ERROR I2, 2I8)
      9 FORMAT  (1H1)
        K4MAX  = 35
        ITMAX  = 4
C
C       READ AND WRITE STATEMENTS
C       READ  (11,  )  READ GENERAL INFORMATION
C       READ  (12,  )  READ COORDINATES
C       WRITE ( 3,  )  LIST OUTPUT
C       WRITE (21,  )  PUNCH OUTPUT
C
C
C     BLOCK A   INITIALIZE THE TRIANGULATION
C
C       READ CODES, FOCAL LENGTH, ETC
        READ (  11, 1) NOTIE,NOPUN,IWT,T1, F,CX,CY,BX1,CE,CR, K2
        STEST = .0008
        IF (T1) 1002, 1002, 1001
   1001 STEST  = T1
   1002 CR     = -CR
        CE     = (CE / 12756. + CR) / (F*F)
        WRITE ( 3, 9)
        IF (K2) 1020, 1020, 1010
C
C       READ LENS CORRECTION TABLE,
C       CHECK SEQUENCE AND NUMBER OF VALUES
   1010 K8     = 1
        DO 1014 K1 = 1,160,8
        K7     = K1 + 7
        READ (  11, 2, END=2910) K3, T3, (CLIST(J), J = K1,K7), K5
        IF (K1-1)  1011, 1011, 1012
```

```
 1011 K4     = K3
      DELR   = T3
      RMAX   = K4 - 1
      RMAX   = RMAX * DELR
 1012 K2     = K2 + 1
      IF (K5 - K2) 2910, 1013, 2910
 1013 IF (K7 - K4) 1014, 1016, 1016
 1014 CONTINUE
      GO TO 2910
C     DIVIDE RADIAL CORRECTION BY INTERVAL
 1016 DO 1017  J = 1,K4
 1017 CLIST(J) = CLIST(J) / (DELR * 1000.)
C
C     READ FIRST CARD OF FIRST MODEL
 1020 K1     = 1
      KA     =1
      GO TO 2010
C
C     INITIALIZE FIRST MODEL
C
 1100 IPC1   = ID2
      MOD    = 1
      K3     = 1
      BY1    = 0.
      GO TO 1200
C
C     INITIALIZE NEXT MODEL
C
 1150 IF (NOTIE) 1151, 1152, 1151
 1151 IF (ID2)  1020, 2999, 1100
 1152 IF (ID2)  1680, 1680, 1160
C
 1160 MOD    = 2
      K55    = K5
      DO 1161  J = 1,K55
      DO 1161  I = 1,4
 1161 LYST(I,J) = LIST(I,J)
      PX     = PX + BX
      PY     = PY + BY
      PZ     = PZ + BZ
      DO 1162  J = 1,9
 1162 T(J+21) = T(J+9)
C
C
C    BLOCK B  PERFORM THE RELATIVE ORIENTATION
C
C     PART 1   FORM NORMAL EQUATIONS
C
 1200 K8     = 3
      K4     = K6
      K      = 1
      ITER   = 0
      KA     = 2
      IPC2   = IPC
```

```
            BY      = BY1
            BZ      = 0.
            DO 1201 J = 1,K4MAX
            DO 1201 I = 1,4
       1201 LIST(I,J) = 0
            WRITE ( 3,55)
            K5      = K4MAX
            IF (K4 - K4MAX) 1202, 1210, 1210
       1202 K5      = K4
            KC      = 1
            IF (K5 - 6) 2950, 1210, 1210
      C
      C     ZERO THE NORMAL EQUATIONS
       1210 DO 1211 M1 = 1,5
            DO 1211 M2 = M1,6
       1211 S(M1,M2) = 0.
            K1      = 0
            ITER    = ITER + 1
            IF (ITER - ITMAX) 1220, 1212, 1212
       1212 K       = 3
       1220 K1      = K1 + 1
            GO TO (2010, 1230, 1230), K
      C
      C     FIRST ITERATION, AFTER READING A POINT
       1221 DO 1222  J = 1,4
       1222 PHC(J,K1)  = W(J)
            X2      = W3
            Y2      = W4
            Z2      = 1.
            GO TO 1250
      C
      C     FOLLOWING ITERATIONS.
      C     USE FIRST GROUP OF ORIENTATION POINTS
       1230 DO 1231  J=1,4
       1231 W(J)    = PHC(J,K1)
            GO TO 2040
      C
      C     LAST ITERATION.
      C     USE REMAINING ORIENTATION POINTS, IF ANY
       1240 K1      = K4MAX + 1
            K7      = K5
       1241 K7      = K7 + 1
            IF (K7-K4) 2010, 2010, 1300
      C
      C     CORRECTION EQUATION FOR ALL THREE ITERATIONS
      C     T6 IS IN SECOND PART OF EQUATION
      C     CROSS PRODUCT B * X1
       1250 T7      = BY - W2 * BZ
            T8      = W1 * BZ - 1.
            T9      = W2 - W1 * BY
      C     CROSS PRODUCT X2 * (B * X1)
            T1      = Y2 * T9 - Z2 * T8
            T2      = Z2 * T7 - X2 * T9
            T3      = X2 * T8 - Y2 * T7
```

```
C           CROSS PRODUCT X1 * X2 AND X1 * X2 . B
            T4      = X2 - W1 * Z2
            T5      = W1 * Y2 - W2 * X2
            T6      = Y2 - W2 * Z2 - T4 * BY - T5 * BZ
            T7      = 1.
            GO TO (1260, 1260, 1251), K
       1251 IF (IWT) 1252, 1260, 1252
C           APPLY WEIGHT
       1252 T8      = W1 * W1 + W2 * W2
            T9      = W3 * W3 + W4 * W4
            T7      = 1. / (2. + 14.*(T8+T9) + 49.*(T8*T8+T9*T9))
C
C           FORM THE NORMAL EQUATIONS
       1260 DO 1261 M1 = 1,5
            T9      = T(M1) * T7
            DO 1261 M2 = M1,6
       1261 S(M1,M2) = S(M1,M2) + T9 * T(M2)
            IF (K1 - K5) 1220, 1262, 1241
       1262 GO TO (1300, 1300, 1263), K
       1263 GO TO (1240, 1300), KC
C
C           PART 2   SOLVE THE NORMAL EQUATIONS
C
C           ELIMINATION
       1300 DO 1305 M1 = 1,5
            M4      = M1 + 1
            DO 1305 M2 = M4,6
            T7      = S(M1,M2) / S(M1,M1)
            IF (M2 - 6) 1303, 1305, 1305
       1303 DO 1304 M3 = M2,6
       1304 S(M2,M3) = S(M2,M3) - T7 * S(M1,M3)
       1305 S(M1,M2) = T7
C           BACK SUBSTITUTION
            DO 1309 M3 = 2,5
            M1      = 6 - M3
            M4      = M1 + 1
            DO 1309 M2 = M4,5
       1309 S(M1,6) = S(M1,6) - S(M1,M2) * S(M2,6)
C
C           ORTHOGONAL MATRIX, COLUMNWISE
            WRITE ( 3, 4) ITER, (S(J,6), J = 1,5)
            W1      = .5 * A1
            W2      = .5 * A2
            W3      = .5 * A3
            T1      = 1. + W1 * W1 - W2 * W2 - W3 * W3
            T5      = 1. - W1 * W1 + W2 * W2 - W3 * W3
            T9      = 1. - W1 * W1 - W2 * W2 + W3 * W3
            T2      = W1 * A2 + A3
            T4      = W1 * A2 - A3
            T3      = W3 * A1 - A2
            T7      = W3 * A1 + A2
            T6      = W2 * A3 + A1
            T8      = W2 * A3 - A1
            W4      = 4. - T1 - T5 - T9
```

```
      DO 1311 J = 1,9
 1311 T(J)    = T(J)  / W4
      BY      = BY + DBY
      BZ      = BZ + DBZ
      M3      = 3
      GO TO (1320, 2080, 2080), K
C
 1320 DO 1321 J = 1,9
 1321 T(J+9) = T(J)
      K       = 2
C
 1330 GO TO (1331, 1331, 1400), K
C     TEST SIZE OF CORRECTIONS
 1331 DO 1332 J = 1,5
      I       = 30. * S(J,6)
      IF (I) 1210, 1332, 1210
 1332 CONTINUE
      K       = 3
      GO TO 1210
C
C
C    BLOCK C   ABSOLUTE ORIENTATION
C
C     FIRST, SCALE THE MODEL
C
 1400 K8      = 4
      KA      = 3
      GO TO (1480, 1401), MOD
 1401 KD      = 1
      BX      = 1.
C
C     STORE SEQ.NO. OF ALL TIES
      M1      = 0
 1411 M1      = M1 + 1
      LYST(5,M1) = 0
      IF (LYST(1,M1)) 1412, 1429, 1412
 1412 DO 1415 M2 = 1,K5
      IF (LIST(1,M2) - LYST(1,M1)) 1413, 1420, 1413
 1413 IF (LIST(1,M2) + LYST(1,M1)) 1415, 1416, 1415
 1415 CONTINUE
      SLIST(M1) = 0.
      GO TO 1429
 1416 SLIST(M1) = 0.
      IF (LYST(1,M1)) 1417, 1417, 1418
 1417 LIST(1,M2) = -LIST(1,M2)
      GO TO 1428
 1418 LYST(1,M1) = -LYST(1,M1)
      GO TO 1428
C
C     REPLACE DISTANCES IN SLIST BY SCALE FACTORS
 1420 DO 1422  J = 1,4
 1422 W(J)    = PHC(J,M2)
C     FOR X2, X1, D, B*X2, LAMBDA1, AND 0.5 LAMBDA3,
      GO TO 2040
```

```
 1423 SLIST(M1) = SLIST(M1) / T6
 1428 LYST(5,M1) = M2
 1429 IF (M1 - K55) 1411, 1440, 1440
C
C      MEAN THE SCALE FACTORS
C
 1440 T1      = 0.
      BX      = 0.
      DO 1443  J = 1,K55
      IF (SLIST(J)) 1442, 1443, 1442
 1442 BX      = BX + SLIST(J)
      T1      = T1 + 1.
 1443 CONTINUE
      IF (T1) 2940, 2940, 1444
 1444 BX      = BX / T1
C      DISCARD ANOMALOUS SCALE FACTORS
      T2      = 0.
      DO 1455  J = 1,K55
      IF (SLIST(J)) 1451, 1455, 1451
 1451 T3      = SLIST(J) - BX
      IF (T3)   1452, 1455, 1453
 1452 T3      = -T3
 1453 IF (T3 - .99999999 * T2) 1455, 1454, 1454
 1454 K7      = J
      T2      = T3
 1455 CONTINUE
      IF (T2 / BX - STEST) 1460, 1460, 1456
 1456 SLIST(K7) = 0.
      LYST(1,K7)= -LYST(1,K7)
      WRITE ( 3,55) LYST(1,K7)
      K7      = LYST(5,K7)
      LIST(1,K7) = -LIST(1,K7)
      GO TO 1440
C
 1460 IF (NOPUN) 1500, 1461, 1500
C      PUNCH UNUSED SCALE TRANSFER POINTS
 1461 DO 1469 J = 1,K55
      ID      = LYST(1,J)
      IF (ID) 1464, 1469, 1462
 1462 IF (LYST(5,J)) 1468, 1468, 1469
 1464 ID      = -ID
 1468 WRITE ( 21, 5) ID3, ID, (LYST(M1,J), M1 = 2,4)
 1469 CONTINUE
      GO TO 1500
C
 1480 BX      = BX1
      PX      = 200000.
      PY      = 400000.
      PZ      = 600000.
C
C      BASE COMPONENTS AND ORIENTATION MATRIX
C
 1500 BY1     = BY
      BY      = BX * BY
```

```
      BZ      = BX * BZ
      GO TO (1510, 1501), MOD
 1501 DO 1502 J = 1,9
 1502 T(J)    = T(J+21)
      M3      = 4
      GO TO 2080
C
C     PRINT MATRIX AND COORDINATES OF CENTRES
C
 1510 DO 1511  J=1,3
 1511 WRITE ( 3, 4) ID1, AR(J,1), AR(J,2), AR(J,3)
      WRITE ( 3,55)
      GO TO (1512, 1513), MOD
 1512 JX      = PX
      JY      = PY
      JZ      = PZ
      J       = 1
      GO TO 1515
 1513 IPC1    = IPC2
      JX      = PX + BX
      JY      = PY + BY
      JZ      = PZ + BZ
      J       = 2
 1515 WRITE ( 3,55)  ID1, IPC1, JX, JY, JZ
      IF (NOPUN) 1529, 1520, 1529
C
C     PUNCH THE ORIENTATION PARAMETERS
 1520 GO TO (1521,1522), J
 1521 M1      = 0
      M2      = 0
      M3      = 0
      M4      = 10000000
      GO TO 1528
 1522 T4      = AR(1,1) + AR(2,2) + AR(3,3) + 1.D0
      T5      = DSQRT(T4)
      M1      = .5D7 * (AR(3,2)-AR(2,3)) / T5
      M2      = .5D7 * (AR(1,3)-AR(3,1)) / T5
      M3      = .5D7 * (AR(2,1)-AR(1,2)) / T5
      M4      = .5D7 * T5
 1528 WRITE ( 21, 5) ID1, IPC1, JX, JY, JZ, M1, M2, M3, M4
 1529 GO TO (1513, 1600), J
C
C
C    BLOCK D   COMPUTE STRIP COORDINATES
C
C     TRIANGULATE STORED ORIENTATION POINTS
C
 1600 KD      = 2
      K1      = 0
 1602 K1      = K1 + 1
      DO 1603  J = 1,4
 1603 W(J)    = PHC(J,K1)
      GO TO 2040
C
```

```
 C      TRIANGULATE ADDITIONAL POINTS
 C
  1610 WRITE ( 3,55)
       GO TO (1611, 1150), KC
  1611 K1      = K4MAX + 1
  1612 GO TO 2010
 C
 C     POSITION VECTOR AND WANT OF INTERSECTION
  1620 IX      = T4 + PX
       IY      = T5 + PY
       IZ      = T6 + PZ
 C     ROUND OFF PROPERLY
       K7      = WANT * DSQRT(T9) + .5
       IF (WANT)  1621, 1630, 1630
  1621 K7      = K7 - 1
 C
 C     LIST ALL POINTS
  1630 ID       = LIST(1,K1)
       IF (NOTIE) 1647, 1631, 1647
  1631 IF (K1-K5) 1632, 1632, 1647
 C     STORE POTENTIAL ORIENTATION POINTS
  1632 SLIST(K1) = 0.
       IF (ID) 1634, 1649, 1633
  1633 SLIST(K1)=AR(1,3)*(T4-BX)+AR(2,3)*(T5-BY)+AR(3,3)*(T6-BZ)
  1634 LIST(2,K1) = IX
       LIST(3,K1) = IY
       LIST(4,K1) = IZ
 C     MEAN COORDINATES OF TIE POINTS
       GO TO (1647, 1641), MOD
  1641 DO 1644  M1 = 1,K55
       IF (K1 - LYST(5,M1)) 1644, 1642, 1644
  1642 IXX     = IX - LYST(2,M1)
       IYY     = IY - LYST(3,M1)
       IZZ     = IZ - LYST(4,M1)
       WRITE ( 3,55) ID1, ID, IX,IY,IZ, K7, IXX,IYY,IZZ
       IF (ID) 1650, 1650, 1643
 C     STORE MEANED COORDINATES OF ACCEPTED TIE POINTS
  1643 LYST(2,M1) = IX - IXX / 2
       LYST(3,M1) = IY -IYY /2
       LYST(4,M1) = IZ - IZZ / 2
       GO TO 1650
  1644 CONTINUE
  1647 IF (ID) 1648, 1649, 1649
  1648 ID      = -ID
  1649 WRITE ( 3,55)  ID1, ID, IX, IY, IZ, K7
 C
 C     PUNCH
  1650 IF (NOPUN) 1679, 1651, 1679
  1651 IF (NOTIE) 1670, 1652, 1670
  1652 IF (K1 - K5) 1602, 1653, 1670
  1653 GO TO (1610, 1660), MOD
 C     PUNCH TIE POINTS WITH PRECEDING MODEL
  1660 DO 1669 J = 1,K55
       IF (LYST(1,J)) 1669, 1669, 1661
```

```
 1661 M2     = LYST(5,J)
      IF (M2) 1669, 1669, 1662
 1662 WRITE ( 21, 5) ID1, (LYST(M1,J), M1 = 1,4)
      LIST(1,M2) = 0
      SLIST(M2) = 0.
 1669 CONTINUE
      GO TO 1610
C     PUNCH POINTS IN PRESENT MODEL
 1670 WRITE ( 21, 5) ID1, ID, IX, IY, IZ
 1679 IF (K1 - K5) 1602, 1610, 1612
C
C     PUNCH ORIENTATION POINTS IN LAST MODEL
 1680 IF (NOPUN) 1689, 1681, 1689
 1681 DO 1684 J = 1,K5
      IF (LIST(1,J)) 1682, 1684, 1683
 1682 LIST(1,J) = -LIST(1,J)
 1683 WRITE ( 21, 5) ID1, (LIST(M1,J), M1 = 1,4)
 1684 CONTINUE
 1689 IF (ID2) 1020, 2999, 2999
C
C
C     SUBROUTINES
C
C     READ A POINT
 2010 READ (  12, 3) ID2, LIST(1,K1), (W(J), J = 1,4), K6
      IF (K1 - 1) 2013, 2011, 2013
 2011 GO TO (2030, 2012, 2012), KA
 2012 ID3    = ID1
      ID1    = ID2
 2013 IF (ID2-ID1)  2030, 2014, 2030
 2014 W1     = (W1 - PP(1)) * CX
      W2     = (W2 - PP(2)) * CY
      W3     = (W3 - PP(3)) * CX
      W4     = (W4 - PP(4)) * CY
C     USE FIRST POINT TO DEFINE POSITION OF PHOTOGRAPH
      GO TO (2016, 2019), K3
 2016 K3     = 2
      FOCAL  = F
      IF (W3-W1) 2017, 2017, 2019
 2017 FOCAL  = -F
 2019 IF (K2) 2025, 2025, 2020
C
C     CORRECT PHOTOGRAPH COORDINATES
 2020 DO 2023 M1 = 1,3,2
      M2     = M1+ 1
      T1     = W(M1)**2 + W(M2)**2
      T2     = DSQRT (T1)  + .0000001
      IF (T2 - RMAX) 2022, 2022, 2920
 2022 J      = T2 / DELR + 1.
      T3     = J
      T3     = DELR * T3 - T2
      T4 = (T3*CLIST(J)+(DELR-T3)*CLIST(J+1))/T2 +CR +CE*T1
      DO 2023  M3 = M1,M2
 2023 W(M3)  = W(M3) + W(M3) * T4
```

```
      2025 DO 2026 J = 1,4
      2026 W(J)    = W(J)  / FOCAL
           GO TO (1221, 2040, 2040), K
C
      2030 IPC     = LIST(1,K1)
           DO 2031  J = 1,4
      2031 PP(J)   = W(J)
           GO TO (1100, 2930, 1150), KA
C
C          VECTOR X2
      2040 DO 2041 J = 1,3
      2041 XR(J)   = AR(J,1)  * W3 + AR(J,2)  * W4  + AR(J,3)
           GO TO (1250, 1250, 2060), KA
C
C          POSITION VECTOR IN STRIP COORDINAIE SYSTEM
C          X = P1 + LAMBDA1 X1 + 0.5 LAMBDA3 D
C          VECTOR X1
      2060 GO TO (2062, 2061), MOD
      2061 GO TO (2062, 2063), KD
      2062 X1      = W1
           Y1      = W2
           Z1      = 1.
           GO TO 2065
      2063 DO 2064 J = 1,3
      2064 XL(J)   = AL(J,1)  * W1 + AL(J,2)  * W2 + AL(J,3)
C          CROSS PRODUCT D = X1 * X2, AND D.D
      2065 T1      = Y1 * Z2 - Z1 * Y2
           T2      = Z1 * X2 - X1 * Z2
           T3      = X1 * Y2 - Y1 * X2
           T9      = T1 * T1 + T2 * T2 + T3 * T3
C          CROSS PRODUCT B * X2
           T4      = BY * Z2 - BZ * Y2
           T5      = BZ * X2 - BX * Z2
           T6      = BX * Y2 - BY * X2
C          LAMBDA1, LAMBDA3, AND 0.5 LAMBDA3
           T7      = (T4 * T1 + T5 * T2 + T6 * T3) / T9
           WANT    = (BX * T1 + BY * T2 + BZ * T3) / T9
           T8      = 0.5 * WANT
C          POSITION VECTOR, REFERRED TO ORIGIN IN FIRST CENTRE
           T4      = T7 * X1 + T8 * T1
           T5      = T7 * Y1 + T8 * T2
           T6      = T7 * Z1 + T8 * T3
           GO TO (1423, 1620), KD
C
C          REPLACE MATRIX AR() BY MATRIX PRODUCT R() * AR()
      2080 DO 2082 M2 = 1,M3
           DO 2081 M1 = 1,3
      2081 W(M1)   = AR(M1,M2)
           DO 2082 M1 = 1,3
      2082 AR(M1,M2) = R(M1,1)*W1 + R(M1,2)*W2 + R(M1,3)*W3
           IF (M3-3) 1330, 1330, 1510
C
C          ERROR MESSAGES
C
```

```
C       ERROR IN LENS CORRECTION CARDS
 2910 K4      = K5
        GO TO 2950
C
 2920 K8      = 2
        J       = K4
        K4      = LIST(1, K1)
        GO TO 2950
 2921 K8      = 3
        K4      = J
        GO TO 2025
C
C       TOO FEW ORIENTATION POINTS    (K8 = 3)
 2930 GO TO (2931, 2999, 2932), K
 2931 KC      = 2
        K5      = K1 - 1
        K4      = K5
        GO TO 2950
 2932 KC      = 3
        K4      = K7 - 1
        GO TO 2950
C
 2934 GO TO (2935, 2937, 1300), KC
 2935 READ (  12,  3) ID2
C
 2936 IF (ID2) 1020, 2999, 2935
 2937 IF (K5 - 6) 2936, 1300, 1300
C       NO SCALE TRANSFER POINT       (K8 = 4)
 2940 MOD     = 1
        NOPUN   = 1
        K4      = ID1
C
 2950 WRITE ( 3, 6) K8, K4
        GO TO (2999, 2921, 2934, 1480), K8
C
 2999 WRITE ( 3,55)
        STOP
        END
```

References

[1] G.H. Schut, Analytical Aerial Triangulation at the National Research Council. Publication AP-PR 7 of the Division of Applied Physics, Ottawa, 1957 (NRC-4672).

[2] - The N.R.C. Program for Analytical Aerial Triangulation on the IBM 1620 and the IBM 650. Publication AP-PR 30, 1964 (NRC-8310).

[3] - An Analysis of Methods in Analytical Aerial Triangulation, including their Mathematical Formulation for Use on Electronic Computers and Practical Results. Publication AP-PR 9, 1957 (NRC-4673); also in Photogrammetria, Vol. XIV, No. 1, 1957-58 and Photogrammetric Engineering, Vol. XXIV, No. 1, 1958.

[4] - Remarks on the Theory of Analytical Aerial Triangulation. Photogrammetria, Vol. XVI, No. 2, 1959-60, and International Archives of Photogrammetry, Vol. XIII, Part 5, 1961.

[5] E. Church, Analytical Computations in Aerial Photogrammetry. Photogrammetric Engineering, Vol. VII, No. 4, 1941. Revised Geometry of the Aerial Photograph - Bulletin of Aerial Photogrammetry No. 15. Syracuse University, 1945.

[6] A.J. McNair, General Review of Analytical Aerotriangulation. Photogrammetric Engineering, Vol. XXIII, No. 3, 1957.

[7] P. Herget, The Reduction of Aerial Photographs on Electronic Computers. Photogrammetric Engineering, Vol. XX, No. 5, 1954.

[8] H.H. Schmid, An Analytical Treatment of the Problem of Triangulation by Stereophotogrammetry. Photogrammetria, Vol. XIII, Nos. 2 and 3, 1956-57.

[9] D.W.G. Arthur, A Stereocomparator Technique for Aerial Triangulation, Ordnance Survey Professional Papers, New Series No. 20, 1955.

[10] E.H. Thompson, A Method of Relative Orientation in Analytical Aerial Triangulation. The Photogrammetric Record, 8, 1956.

[11] G.H. Schut, A FORTRAN Program for the Adjustment of Strips and of Blocks by Polynomial Transformations. Publication AP-PR 33, Second Edition, 1968 (NRC-9265).

[12] Anon. Standard Atmosphere - Tables and Data for Altitudes up to 65800 feet. NACA Report 1235, 1955.

[13] R.A. Minzner, K.S.W. Champion, and H.L. Pond, The ARDC Model Atmosphere, 1959. Air Force Surveys in Geophysics No. 115, Air Force Cambridge Research Center, 1959.

[14] Anon. U.S. Standard Atmosphere, 1962. U.S. Government Printing Office, 1962.

[15] S. Bertram, Atmospheric Refraction. Photogrammetric Engineering, Vol. XXXII, No. 1, 1966.

[16] B. Edlén, The Dispersion of Standard Air. Journal of the Optical Society of America, Vol. 43, 1953, No. 5.

[17] G.W.C. Kaye and T.H. Laby, Physical and Chemical Constants, Xth edition. Longmans, Green, and Co., London, 1948.

[18] A. Leyonhufvud, On Photogrammetric Refraction. Photogrammetria, Vol. IX, No. 3, 1952-53.

[19] G.H. Schut, Construction of Orthogonal Matrices and Their Application in Analytical Photogrammetry. Photogrammetria, XV~~II~~, No. 4, ~~1960-61~~.

[20] B. Hallert, Investigations of the Weights of Image Coordinates in Aerial Photographs. Photogrammetric Engineering, Vol. XXVII, No. 4, 1961.

[21] E.H. Thompson, A Rational Algebraic Formulation of the Problem of Relative Orientation. Photogrammetric Record, Vol. III, No. 14, 1959.

[22] - The St. Faith Experiment. Part I. Report of Proceedings of the Conference of Commonwealth Survey Officers, 1963, Part I. Her Majesty's Stationery Office, London, 1964.

[23] Anon. U.S. Standard Atmosphere Supplements, 1966. U.S. Government Printing Office, 1967.

[24] G.H. Schut, Photogrammetric Refraction. Photogrammetric Engineering, Vol. 35, No. 1, 1969.

[25] L. Brand, *Vector and Tensor Analysis*. John Wiley and Sons, Inc., New York (1947).